

MODERN APPROACHES IN GEOPHYSICS

ROBERTO SABADINI AND BERT VERMEERSEN

Global Dynamics of the Earth

**Applications of Normal Mode Relaxation
Theory to Solid-Earth Geophysics**



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GLOBAL DYNAMICS OF THE EARTH

MODERN APPROACHES IN GEOPHYSICS

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GLOBAL DYNAMICS OF THE EARTH

Applications of Normal Mode Relaxation
Theory to Solid-Earth Geophysics

by

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To Giulia and Armanda
R. S.

Contents

Acknowledgments	xi
Foreword	xiii
1. NORMAL MODE THEORY IN VISCOELASTICITY	1
1. Rheological Models	1
2. Momentum and Poisson Equations	3
3. Expansion in Spherical Harmonics: Spheroidal and Toroidal Solutions	8
3.1 Spheroidal Solution for the Incompressible Case	10
3.2 Toroidal Solution for the Incompressible Case	16
4. Fundamental Spheroidal Matrix in the Laplace Domain	17
5. Propagator Matrix Technique	18
5.1 Propagation of the Spheroidal Solution	19
5.2 Propagation of the Toroidal Solution	23
6. Inverse Relaxation Times for Incompressible Earth Models	24
7. Phase-change Interface	26
8. Loading the Earth	29
8.1 Internal Loading: Earthquakes, Subduction	30
8.2 Surface Loading: Point Mass and Tidal Forcing	31
8.3 Solving for the Displacement and Perturbation in the Gravitational Potential	32
9. Approximation Method for High-Degree Harmonics	35
9.1 Approximation of the Solution	36
9.2 Rescaling the Solution	39
2. MULTI-LAYER MODELS	45
1. Analytical Development of Stratified Viscoelastic Spherical Earth Models	45
1.1 Uniform Mantle Viscosity and Elastic Lithosphere	52
1.2 Convex Mantle Viscosity and Elastic Lithosphere	58
1.3 Convex Mantle Viscosity and Viscoelastic Lithosphere	71
2. Shallow Inversions in PREM Density Stratification	74
3. Volume-averaged vs. Fixed-boundary Contrast	77

4.	Isolation Functions	81
3.	ROTATIONAL DYNAMICS OF VISCOELASTIC PLANETS	99
1.	Introduction to Earth Rotation	99
2.	Rotational Changes for a Rigid Earth	103
3.	Adjustment of the Equatorial Bulge	108
4.	Developments of Linearized Rotation Theories	110
4.1	Comparison Between Different Rotation Theories	115
4.2	Omission of the M_0 Rotation Mode	116
4.3	Analytical Formula for the M_0 Rotation Mode	119
4.4	Unification of the Different Approaches	122
5.	Long-term Behavior of the Rotation Equation	123
4.	POLAR WANDER AND $\dot{J}_2$ INDUCED BY ICE-SHEET LOADING	127
1.	The Concept of True Polar Wander (TPW)	127
2.	The Inference of Mantle Viscosity From TPW and $\dot{J}_2$ Data	130
3.	Loading	132
4.	Mantle Viscosity	136
4.1	Variations in Depth of the Two-Layer Mantle Viscosity Profile	140
4.2	Upper Mantle Viscosities Lower Than 10^{21} Pa s	144
5.	Inference of Lithospheric Thickness From TPW and Length of Day Variations	149
6.	Ice Age Cycles and the Polar Wander Path	150
7.	Influences of Phase-Change vs. Chemical Stratification	156
8.	Trade-off Between Mantle Viscosity and Density Contrasts	165
9.	Impact of Mantle Stratification on TPW During the Ice Ages	168
5.	DETECTION OF THE TIME-DEPENDENT GRAVITY FIELD AND GLOBAL CHANGE	173
1.	Changes in the Long-Wavelength Geoid Components from Satellite Laser Ranging Techniques	173
2.	Trade-off between Lower Mantle Viscosity and Present-Day Mass Imbalance in Antarctica and Greenland	179
6.	SEA-LEVEL CHANGES	189
1.	Introduction	189
2.	Sea-level Variations, Geoid and Gravity Anomalies Due to Pleistocene Deglaciation	192
2.1	Mathematical Formulation	192
2.2	Sea-Level Variations, the Geoid and Free-Air Gravity Anomalies	195
3.	Glacial Isostatic Adjustment (GIA) vs. Tectonic Processes: the Example of the Mediterranean Sea	202

4.	Sea-Level Fluctuations Induced by Polar Wander	210
5.	Sea-Level Changes Induced by Subduction	216
5.1	Sea-Level Variations, Geoid Anomalies and the Long-Wavelength Dynamic Topography	218
5.2	A Single Sinking Slab	219
5.3	A Distribution of Slabs	222
7.	TECTONICS AND THE ROTATION OF THE EARTH AND TERRESTRIAL PLANETS	227
1.	Mountain Building and Earth Rotation	227
1.1	Uplift Histories	229
1.2	Changes in J_2 and Linear Root Formation History	231
1.3	Polar Wander and Heaviside, Linear and Exponential Root Formation History	234
2.	Subduction Effects on TPW	249
2.1	Non-linear Rotation Theory	249
2.2	Polar Wander Velocity for a Distribution of Slabs	257
3.	Polar Wander on the Earth, Moon, Mars and Venus	258
8.	POST-SEISMIC DEFORMATION	269
1.	Global Post-seismic Deformation	269
2.	Post-seismic Deformation for Shallow Earthquakes	276
2.1	The Umbria-Marche (1997) Earthquake	278
2.2	The Irpinia (1980) Earthquake	286
	Appendices	293
	A– Fundamental Matrix for the Compressible Case	293
	B– Complex Contour Integration	299
	1. Analytical Functions	299
	2. Cauchy Integral Representation	300
	3. Residue Theorem	302
	C– Forcing Terms for a Dislocation Source	305
	1. Strike-slip	305
	1.1 Spheroidal Components	305
	1.2 Toroidal Components	306
	2. Dip-slip	306
	2.1 Spheroidal Components	306
	2.2 Toroidal Components	307
	References	309
	Index	325

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Foreword

With this book we provide graduate students with the mathematical techniques necessary to solve some basic problems related to the deformation of the Earth by a variety of geophysical and geological processes involving the rheological behavior of the mantle. We deal with global processes involving the deformation of the Earth as a whole, as in rotation dynamics, or, on a smaller scale, with the response of the planet to the redistribution of surface loads and seismic sources associated with earthquakes faulting the lithosphere.

When we had the idea for this book, our major concern was to write something that could be used by students to write their own codes for solving problems in linear viscoelasticity and, at the same time, to tell them what we have learned in rotation dynamics and related processes like induced sea-level changes.

Now we realize that we can go beyond the original scope of our work and consider it as a tool for gaining in-depth physical insight into the complex relations among the various processes involving our planet, from surface and deep mantle processes to rotation instabilities and, eventually, long time scale climate variabilities.

In the following decades, major advancements in our understanding of the dynamics and evolution of the Earth as an integrated system, including a variety of geophysical processes that affect the life of human beings, will come from joint efforts in geophysics and geodesy, from a tight link between the mathematical modeling of geophysical phenomena and the constraints provided by the gravity field, of its time dependent and static components at long and short wavelengths, and by the geodetically detected crustal deformation. Our book provides the mathematical formulation to establish the link between geophysics and geodesy.

Our book opens up new perspectives on the physics of the Earth's interior, looking at our planet in an integrated fashion and focusing on the geophysical and geodetic techniques that record, over a broad spectrum of spatial

wavelengths, the ongoing modifications in the shape and gravity of the Earth. Basic issues related to the rheological properties of the Earth's mantle and to its slow deformation will be understood, within the framework of an analytical normal mode theory; fundamentals of this theory are developed in the first, tutorial part, while a wide range of applications are considered in the second part.

In the study of the physics of the Earth's interior, this book bridges the gap between seismology and geodynamics.

Our hope is of course that not only graduate students but also researchers working in the fields touched upon by this book may find it useful for their own work and teaching.

Roberto Sabadini and Bert Vermeersen,
Milan and Delft, December 2003

Chapter 1

NORMAL MODE THEORY IN VISCOELASTICITY

1. RHEOLOGICAL MODELS

In modeling a particular geophysical phenomenon, the choice of the rheology used depends on 1) mathematical difficulty, 2) the quality of the geophysical data which the calculations of the model are required to match and 3) our knowledge of the rheological behavior of the medium at hand. Over the last few decades a considerable amount of knowledge has been gained about mantle rheology in terms of the values of rheological parameters and deformation mechanisms. For instance, what is most important, as far as mantle convection is concerned, is clearly the strong temperature dependence of the viscosity which the laboratory-derived values of the activation energy and volume seem to suggest. This intense interest in understanding convection in a fluid with markedly temperature-dependent viscosity is attested by the recent fundamental studies by geophysicists using analytical, numerical and experimental methods. In what follows, however, rather than discussing topics of mantle rheology and mantle convection, for which we refer to the book by Ranalli (1995), we will try to address the main questions that are at issue in attempting to study transient and long time scale geodynamic phenomena in a wide arc of time scales, ranging from years, characteristic of post-seismic deformation, to hundreds of millions of years as in the case of true polar wander driven by subduction, making use of the analytical normal mode theory in viscoelasticity with different models of mantle rheology. In Figure 1.1 we sketch the entire geodynamic spectrum spanning the whole range of phenomenological time scales. One of the key questions is whether one can devise a constitutive law which can satisfactorily model all these phenomena, from the anelastic transient regime to the steady-state domain.

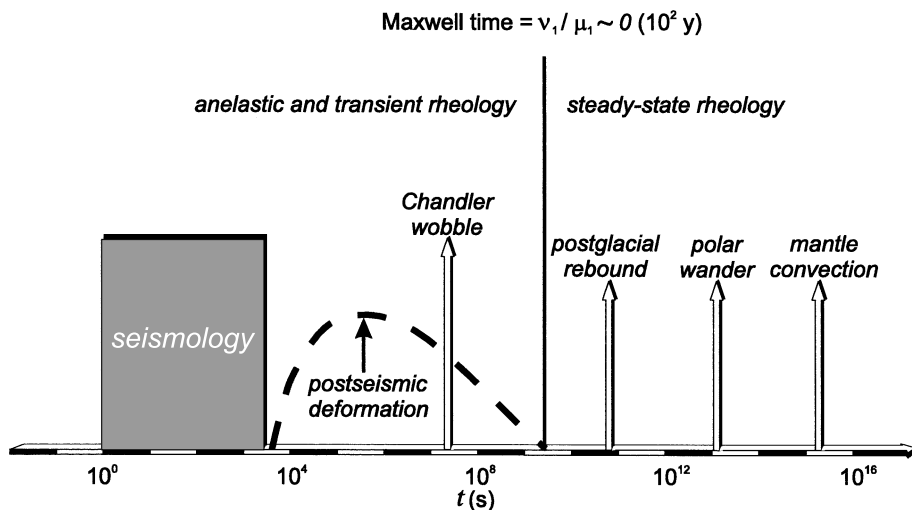


Figure 1.1. Diagram illustrating the relation of the characteristic time scale for several geophysical phenomena to the Maxwell time of the mantle defined as ν_1 / μ_1 - with ν_1 and μ_1 denoting respectively the steady state mantle viscosity and rigidity - which separates the steady state and the transient regimes of mantle creep.

The appropriate constitutive relation which is to be employed in analyzing transient geodynamic phenomena, such as postglacial rebound or Glacial Isostatic Adjustment (GIA), is currently a matter of controversy in geophysics. Advocates of non-linear rheology (e.g., Melosh, 1980) use as supporting arguments the laboratory data of single-crystal olivine whose power law index is about three (Goetze, 1978; Durham and Goetze, 1977). But there is now mounting evidence that at the stress levels in postglacial rebound (less than $0(10^2 \text{ bar})$) the creep mechanism may in fact be linear for polycrystalline aggregates (Relandeau, 1981) since grain boundary processes, such as Coble creep, may become dominant. There are also recent theoretical studies indicating that the power law index changes gradually with stress and hence the transition stress which marks the boundary between linear and nonlinear behavior is not as sharply defined as has previously been thought (Greenwood *et al.*, 1980). Indeed, a proper mathematical formulation of the mixed initial and boundary-value problems associated with nonlinear viscoelasticity is a formidable one, fraught with numerical difficulties. It is also important to note that there is no unambiguous evidence in either the postglacial rebound event or in other types of geodynamic data which absolutely requires a nonlinear viscoelastic rheology, in spite of claims to the contrary. For these reasons, geophysicists tend to prefer the simple linear models in viscoelasticity, which allow for a considerably simpler mathematical treatment of the dynamics. Moreover, the

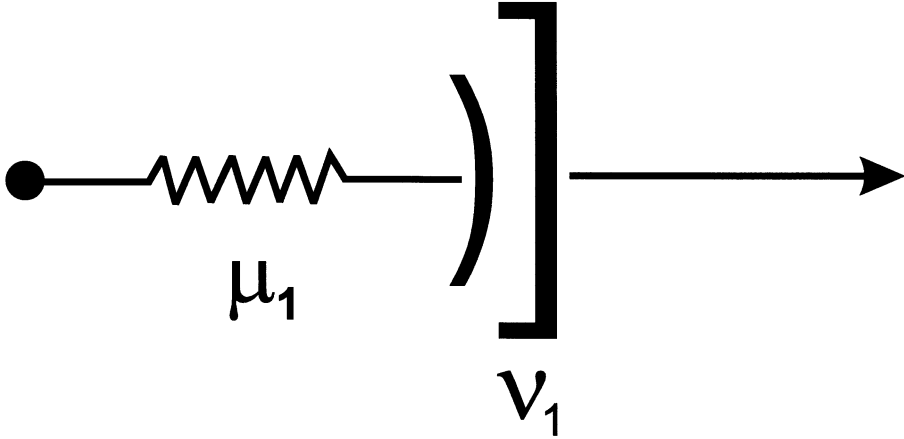


Figure 1.2. Mechanical analog of Maxwell rheology. The elastic response is governed by the shear modulus μ_1 of the spring while the long-term creep is controlled by the viscosity ν_1 of the dashpot.

linear approach also allows one to study easily the potentially interesting effects of the interaction between transient and steady-state rheologies. The simplest viscoelastic model which can describe the Earth as an elastic body for short time scales and as a viscous fluid for time scales characteristic of continental drift is that of a linear Maxwell solid. Figure 1.2 shows a standard one-dimensional spring and dashpot analog of the Maxwell rheology. The speed for shear wave propagation depends on the square root of the instantaneous rigidity μ_1 , whereas the strength of mantle convection depends inversely upon the magnitude of the steady-state viscosity ν_1 .

A powerful method of solving transient problems of linear viscoelasticity has been the use of the Correspondence Principle (Peltier, 1974), which allows one to employ the elastic solution of a given problem in the Laplace-transformed version of the corresponding viscoelastic problem. The Correspondence Principle for the Maxwell rheology and normal mode theory is introduced hereafter in this chapter.

2. MOMENTUM AND POISSON EQUATIONS

The following mathematical model describes the response of the viscoelastic linear Maxwell earth model to a delta function type of force. After having derived the Green functions, the response of the Earth to arbitrary loads or forces in space and time is found by convolving these functions with the loads or forces.

We assume that the rheological laws (stress - strain and stress - strain rate relations) are linear and that the strains are infinitesimal. We do not deal with non-linear rheologies and finite strain theory, but that does not imply that these are not important for the Earth Sciences. However, for a wide spectrum of solid-Earth relaxation processes, we can neglect both.

For long time scale processes the inertial forces vanish, and conservation of linear momentum requires that the body forces $\mathbf{F}$ per unit mass acting on the element of the body are balanced by the stresses that act on the surface of the element. At any instant of time we thus have for the stress tensor $\boldsymbol{\sigma}$ acting on the infinitesimal block with density ρ :

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{F} = 0. \quad (1.1)$$

We assume first that the Earth is compressible, laterally homogeneous (but radially stratified!) and hydrostatically pre-stressed. We also assume that the Earth is not rotating (we will study rotation at a later stage). We will consider the elastic equations of motion, since any linear viscoelastic problem, which is of interest to us, is equivalent to an elastic problem in the Laplace domain, in agreement with the Correspondence Principle, as will be shown in the following. We thus solve the momentum and gravity equations for an elastic medium, and only at the last stage, once the elastic solution has been obtained, the Correspondence Principle is applied.

The stress tensor $\boldsymbol{\sigma}$ is the sum of the initial pressure, due to the hydrostatically prestressed conditions, plus a perturbation $\boldsymbol{\sigma}_1$, so that $\boldsymbol{\sigma}$ reads

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - p_0 \mathbf{I}. \quad (1.2)$$

$\boldsymbol{\sigma}_1$ denotes a tensor which describes the acquired, non-hydrostatic stress, which will be related to the strain by means of the appropriate constitutive equations. The hydrostatic pressure p_0 , with $\mathbf{I}$ the identity matrix, enters the equation above with the minus sign since it denotes a compressive stress, which is negative according to the convention that stresses are positive when they act in the same direction as the outward normal to the surface. On the elementary surface enclosing the elementary volume in which the equation of equilibrium holds, the stress due to the load of the overlying material, namely the pressure, is negative according to this convention. The equation of conservation of linear momentum thus reads

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla p_0 + \rho \mathbf{F} = 0. \quad (1.3)$$

If the body is subject to an elastic displacement $\mathbf{u}$ in t_0 , then the pressure in $t_0 + \delta t$ at a fixed point in space is given by

$$p_0(t_0 + \delta t) = p_0(t_0) - \mathbf{u} \cdot \nabla p_0. \quad (1.4)$$

The minus sign accounts for the fact that the pressure has to increase at a fixed point in space if the elastic displacement occurs in the opposite direction with respect to the pressure gradient.

The equation of conservation of linear momentum after the elastic displacement reads, with $p_0(t_0 + \delta t)$ instead of $p_0(t_0)$,

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla p_0(t_0) + \nabla(\mathbf{u} \cdot \nabla p_0) + \rho \mathbf{F} = 0. \quad (1.5)$$

The gradient of the initial pressure is given by

$$\nabla p_0 = -\rho_0 g \hat{\mathbf{e}}_r, \quad (1.6)$$

where $\hat{\mathbf{e}}_r$ denotes the unit vector, positive outward from the Earth center. With this explicit expression of the gradient of the initial pressure, the equation of equilibrium becomes

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla p_0(t_0) - \nabla(\rho_0 g \mathbf{u} \cdot \hat{\mathbf{e}}_r) + \rho \mathbf{F} = 0. \quad (1.7)$$

The force $\mathbf{F}$ can generally be split into gravity and all kinds of other forcings and loads (e.g., tidal forces, centrifugal forces, loads due to ice-water redistribution, earthquake forcings, etc.). Let us, for the moment, assume that the force $\mathbf{F}$ is gravity (so essentially the condition of a free, self-gravitating Earth with no other forcings or loads acting on its surface or interior) and that, as it is a conservative force, it can be expressed as the negative gradient of the potential field ϕ

$$\mathbf{F} = -\nabla \phi. \quad (1.8)$$

The potential field ϕ can be written as

$$\phi = \phi_0 + \phi_1, \quad (1.9)$$

with ϕ_0 as the field in the initial state and ϕ_1 the infinitesimal perturbation.

The linearized equation of momentum becomes, with ρ_1 the perturbation in the density and g the gravity,

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla(\rho_0 g \mathbf{u} \cdot \hat{\mathbf{e}}_r) - \rho_0 \nabla \phi_1 - \rho_1 g \hat{\mathbf{e}}_r = 0, \quad (1.10)$$

since the second term of equation (1.7) is canceled by the term $-\rho_0 \nabla \phi_0$.

The first term of equation (1.10) describes the contribution from the stress, the second term the advection of the (hydrostatic) pre-stress, the third term the changed gravity (self-gravitation) and the fourth term the changed density (compressibility). In cases where self-gravitation is neglected, the third term will be zero, while in the case of incompressibility the fourth term will be zero.

The perturbed gravitational potential ϕ_1 satisfies the Poisson equation

$$\nabla^2 \phi_1 = 4\pi G \rho_1, \quad (1.11)$$

with G as the universal gravitational constant. In the case of incompressibility the right-hand term will be zero since $\rho_1 = 0$, and equation (1.11) reduces to the Laplace equation

$$\nabla^2 \phi_1 = 0. \quad (1.12)$$

The equations above need to be supplemented by a constitutive equation describing how stress and strain (or strain rate) are related to each other. Throughout this book we will make use of the Maxwell model depicted in Figure 1.2. The momentum and Poisson equations for an elastic solid will first be expanded in spherical harmonics and only afterwards the Correspondence Principle will be applied in order to retrieve the viscoelastic solution, which in our case is specialized for an incompressible material.

The equilibrium and Poisson equations can be written in spherical coordinates with r denoting the radial distance from the center of the Earth, θ the colatitude and ϕ the longitude (Schubert *et al.*, 2001, page 281), with derivatives with respect to r and θ denoted by ∂_r and ∂_θ . By assuming that there are no longitudinal components in the fields as well as in their derivatives, with symmetric deformation around the polar axis and taking account of the continuity equation written as follows

$$\rho_1 = -\nabla \cdot (\rho_0 \mathbf{u}) = -\mathbf{u} \cdot \hat{\mathbf{e}}_r \partial_r \rho_0 - \rho_0 \nabla \cdot \mathbf{u}, \quad (1.13)$$

with $\Delta = \nabla \cdot \mathbf{u}$ and ρ_1 denoting the perturbed density, the two r and θ components of the momentum and Poisson equations become

$$\begin{aligned} & -\rho_0 \partial_r \phi_1 + \rho_0 g_0 \Delta - \rho_0 \partial_r (u g_0) + \partial_r \sigma_{rr} \\ & + r^{-1} \partial_\theta \sigma_{r\theta} + r^{-1} (2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta} \cot \theta) = 0 \end{aligned} \quad (1.14)$$

$$\begin{aligned} & -\rho_0 r^{-1} \partial_\theta \phi_1 - \rho_0 g_0 r^{-1} \partial_\theta u \\ & + \partial_r \sigma_{r\theta} + r^{-1} \partial_\theta \sigma_{\theta\theta} + r^{-1} ((\sigma_{\theta\theta} - \sigma_{\phi\phi}) \cot \theta + 3\sigma_{r\theta}) = 0 \end{aligned} \quad (1.15)$$

$$r^{-2} \partial_r (r^2 \partial_r \phi_1) + (r^2 \sin \theta)^{-1} \partial_\theta (\sin \theta \partial_\theta \phi_1) = -4\pi G (\rho_0 \Delta + u \partial_r \rho_0), \quad (1.16)$$

where σ_{rr} , $\sigma_{\theta\theta}$, $\sigma_{\phi\phi}$ and $\sigma_{r\theta}$ denote the stress components in spherical coordinates expressed as

$$\sigma_{rr} = \lambda \Delta + 2\mu \epsilon_{rr} \quad (1.17)$$

$$\sigma_{\theta\theta} = \lambda\Delta + 2\mu\epsilon_{\theta\theta} \quad (1.18)$$

$$\sigma_{\phi\phi} = \lambda\Delta + 2\mu\epsilon_{\phi\phi} \quad (1.19)$$

$$\sigma_{r\theta} = 2\mu\epsilon_{r\theta}, \quad (1.20)$$

expressed in terms of the strain tensor components ϵ_{rr} , $\epsilon_{\theta\theta}$, $\epsilon_{\phi\phi}$, $\epsilon_{r\theta}$ and Lamé parameters λ and μ .

In this section we focus on the spheroidal components with the ϕ component of the displacement equal to zero. The strain tensor components are thus given in terms of the displacement

$$\epsilon_{rr} = \partial_r u \quad (1.21)$$

$$\epsilon_{\theta\theta} = r^{-1}(\partial_\theta v + u) \quad (1.22)$$

$$\epsilon_{\phi\phi} = r^{-1}(v \cot \theta + u) \quad (1.23)$$

$$\epsilon_{r\theta} = \frac{1}{2}(\partial_r v - r^{-1}v + r^{-1}\partial_\theta u), \quad (1.24)$$

where u and v denote the radial and tangential (along meridian) components of the displacement vector.

The r component of the momentum equations becomes in terms of the displacement components

$$\begin{aligned} & -\rho_0 \partial_r \phi_1 + \rho_0 g \Delta - \rho_0 \partial_r (ug) + \partial_r (\lambda \Delta + 2\mu \partial_r u) \\ & + \frac{\mu}{r^2} [4r \partial_r u - 4u + r(\partial_\theta \partial_r v + \partial_r v \cot \theta) \\ & + \partial_\theta^2 u + \partial_\theta u \cot \theta - 3(\partial_\theta v + v \cot \theta)] = 0. \end{aligned} \quad (1.25)$$

The θ component of the momentum equation is given by

$$\begin{aligned} & -(\frac{\rho_0}{r}) \partial_\theta \phi_1 - (\frac{\rho_0 g}{r}) \partial_\theta u + \mu \partial_r (\partial_r v - \frac{v}{r} + \frac{1}{r} \partial_\theta u) \\ & + \frac{1}{r} \partial_\theta (\lambda \Delta) + \frac{2\mu}{r^2} (\partial_\theta^2 v + \partial_\theta v \cot \theta - v \cot^2 \theta - v) \\ & + \frac{3\mu}{r} \partial_r v + \frac{5\mu}{r^2} \partial_\theta u - \frac{\mu}{r^2} v = 0. \end{aligned} \quad (1.26)$$

The terms in equations (1.25) and (1.26) have been arranged in such a way as to make it easier the elimination of the derivatives with respect to the coordinate θ by making use of the Legendre equations and of its derivatives, once the fields u and v are expanded in Legendre polynomials, as done in section 1.3.

From equation (A.123) by Ben-Menahem and Singh (1981) we obtain the divergence of the displacement in spherical coordinates, which will be expanded in Legendre polynomials in the following section 1.3,

$$\nabla \cdot \mathbf{u} = \partial_r u + \frac{2}{r}u + \frac{1}{r}\partial_\theta v + \frac{\cot\theta}{r}v. \quad (1.27)$$

3. EXPANSION IN SPHERICAL HARMONICS: SPHEROIDAL AND TOROIDAL SOLUTIONS

In principle, deformation, stress field and gravity field can be solved by means of numerical integration techniques from the three equations (1.16) and (1.25)-(1.26). However, we will see that it is also possible to solve these equations virtually analytically by means of normal mode modeling in the Laplace-transformed domain, as stated by the Correspondence Principle. This kind of analytical solution has a few great advantages: it leads us to a deeper insight into the mechanisms of the relaxation process with additional checking possibilities, and certainly for spherical (global) models it often proves easier to be used than numerical integration techniques. Numerical integration techniques also have their advantages. For instance, they can generally deal more easily with more elaborate models (e.g., those that use non-linear rheologies or lateral variations) and often prove simpler to be used in half-space (regional) models, although also for half-space models analytical solutions are available (e.g., Wolf, 1985a,b). So the numerical and analytical models are to be seen as being more complementary than redundant.

Following our normal mode approach, the momentum equations (1.25)-(1.26) and the Poisson equation (1.16) must be expanded in spherical harmonics. Phinney and Burridge (1973) provide the methodology to expand any tensor field in spherical harmonics, defined by

$$Y_l^m(\theta, \phi) = (-1)^m P_l^m(\cos\theta) \exp(im\phi), \quad (1.28)$$

with $l = 0, 1, 2, \dots$ and $m = -l, -l + 1, \dots, l$ and $P_l^m(\cos\theta)$ denoting the associated Legendre function

$$P_l^m(\cos\theta) = \frac{(1 - \cos^2\theta)^{m/2}}{2^l l!} \frac{d^{l+m}}{d(\cos\theta)^{l+m}} (\cos^2\theta - 1)^l. \quad (1.29)$$

Since our earth model is laterally homogeneous, the spheroidal solution does not contain any longitudinal component, and the spheroidal radial and tangential displacement components, the divergence Δ and the perturbation of the gravitational potential that will be used throughout are expanded in Legendre polynomials $P_l(\cos\theta)$ rather than in spherical harmonics $Y_l^m(\theta, \phi)$. We thus

have for the spheroidal part the radial displacement u , the tangential (colatitudinal) component v , Δ and the perturbation in the gravitational potential ϕ_1 as functions of the scalars U_l , V_l , χ_l and ϕ_l , which depend solely on the harmonic degree l and on the radial distance r from the center of the Earth

$$u = \sum_{l=0}^{\infty} U_l(r) P_l^m(\cos \theta), \quad (1.30)$$

$$v = \sum_{l=0}^{\infty} V_l(r) \partial_{\theta} P_l(\cos \theta), \quad (1.31)$$

$$\Delta = \sum_{l=0}^{\infty} \chi_l(r) P_l(\cos \theta), \quad (1.32)$$

$$\phi_1 = - \sum_{l=0}^{\infty} \phi_l(r) P_l(\cos \theta), \quad (1.33)$$

where the Legendre polynomial $P_l(\cos \theta)$ is obtained from the Rodrigues' formula

$$P_l(\cos \theta) = \frac{1}{2^l l!} \frac{d^l}{d(\cos \theta)^l} (\cos^2 \theta - 1)^l, \quad (1.34)$$

or from $m = 0$ in the previous definition (1.29) of the associated Legendre function. Note that the spheroidal solution does not carry any longitudinal displacement, as anticipated above.

The toroidal displacement components, v' along colatitude and w along longitude, are defined by the following expansion in spherical harmonics

$$v' = \sum_{l=0}^{\infty} W_l(r) \sum_{m=-l}^l \nabla_{\phi} Y_l^m(\theta, \phi) \quad (1.35)$$

$$w = - \sum_{l=0}^{\infty} W_l(r) \sum_{m=-l}^l \nabla_{\theta} Y_l^m(\theta, \phi), \quad (1.36)$$

which provide the toroidal components of the latitudinal and longitudinal displacements v' and w as a function of the W_l scalar harmonic coefficients. Collectively, U_l , V_l , W_l and ϕ_l are named scalar eigenfunctions and satisfy linear systems of homogeneous ordinary differential equations in the radial variable r .

3.1 SPHEROIDAL SOLUTION FOR THE INCOMPRESSIBLE CASE

By making use of the expansion of the spheroidal displacement components u, v, Δ and ϕ_1 defined in equations (1.30)-(1.33), of the Legendre equation

$$\frac{d^2}{d\theta^2} P_l(\theta) + \cot \theta \frac{d}{d\theta} P_l(\theta) = -l(l+1)P_l(\theta), \quad (1.37)$$

and of the derivative of the Legendre equation

$$\frac{d^3}{d\theta^3} P_l + \frac{d^2}{d\theta^2} P_l \cot \theta - \frac{d}{d\theta} P_l [1 + \cot^2 \theta] = -l(l+1) \frac{d}{d\theta} P_l, \quad (1.38)$$

the r and θ components of the momentum equations become for each harmonic degree l

$$\begin{aligned} & \rho_0 \partial_r \phi_l + \rho_0 g \chi_l - \rho_0 \partial_r (g U_l) + \partial_r (\lambda \chi_l + 2\mu \partial_r U_l) \\ & + r^{-2} \mu [4r \partial_r U_l - 4U_l + l(l+1)(-U_l - r \partial_r V_l + 3V_l)] = 0 \end{aligned} \quad (1.39)$$

$$\begin{aligned} & \rho_0 \phi_l - \rho_0 g U_l + \lambda \chi_l + r \mu \partial_r (\partial_r V_l - r^{-1} V_l + r^{-1} U_l) \\ & + r^{-1} \mu [5U_l + 3r \partial_r V_l - V_l - 2l(l+1)V_l] = 0. \end{aligned} \quad (1.40)$$

The Poisson equation becomes

$$\partial_r^2 \phi_l + \frac{2}{r} \partial_r \phi_l - \frac{l(l+1)}{r^2} \phi_l = 4\pi G (\rho_0 \chi_l + U_l \partial_r \rho_0). \quad (1.41)$$

The terms in the θ component of the momentum equation have been multiplied by r . The r and θ components of the momentum equations are expanded respectively on the Legendre polynomial P_l and on its derivative $\partial_\theta P_l$.

Once equations (1.39)-(1.41) are solved for each degree l , we obtain the fields summing up all the linearly independent solutions via equations (1.30)-(1.33).

For the divergence of the displacement Δ we obtain, by making use of equation (1.27) and of the Legendre equation,

$$\chi_l = \partial_r U_l + 2r^{-1} U_l - l(l+1)r^{-1} V_l. \quad (1.42)$$

The spheroidal solution vector $\mathbf{y}$ is defined as (Wu and Peltier, 1982)

$$\begin{aligned}
 y_1 &= U_l \\
 y_2 &= V_l \\
 y_3 &= \Pi_l + 2\mu\partial_r U_l \\
 y_4 &= \mu(\partial_r V_l - \frac{V_l}{r} + \frac{U_l}{r}) \\
 y_5 &= -\phi_l \\
 y_6 &= -\partial_r \phi_l - \frac{(l+1)}{r}\phi_l + 4\pi G\rho_0 U_l,
 \end{aligned} \tag{1.43}$$

where $\Pi_l = \lambda\chi_l$. The quantity y_6 is for obvious reasons sometimes nicknamed the potential stress. Why y_6 is chosen rather than $\partial_r \phi_l$ will become clear when the boundary conditions are discussed as follows.

From the condition of incompressibility

$$\chi_l = 0 \tag{1.44}$$

and homogeneity of each layer

$$\partial_r \rho_0 = 0, \tag{1.45}$$

we obtain the Laplace equation

$$\partial_r^2 \phi_l + \frac{2}{r}\partial_r \phi_l - \frac{l(l+1)}{r^2}\phi_l = 0 \tag{1.46}$$

and momentum equations

$$\begin{aligned}
 &\rho_0 \partial_r \phi_l - \rho_0 \partial_r (gU_l) + \partial_r (\Pi_l + 2\mu\partial_r U_l) \\
 &+ \frac{\mu}{r^2} [4r\partial_r U_l - 4U_l + l(l+1)(-U_l - r\partial_r V_l + 3V_l)] = 0
 \end{aligned} \tag{1.47}$$

$$\begin{aligned}
 &\rho_0 \phi_l - \rho_0 gU_l + \Pi_l + \mu r \partial_r (\partial_r V_l - \frac{V_l}{r} + \frac{U_l}{r}) \\
 &+ \frac{\mu}{r} [5U_l + 3r\partial_r V_l - V_l - 2l(l+1)V_l] = 0,
 \end{aligned} \tag{1.48}$$

where we have taken into account that the product $\lambda\chi_l$ remains finite for an incompressible body. From $\chi_l = 0$ we obtain the following relationship between the tangential and radial components of the displacement

$$V_l = \frac{r\partial_r U_l + 2U_l}{l(l+1)} \tag{1.49}$$

once we make use of the condition of incompressibility $\nabla \cdot \mathbf{u} = 0$.

Exercise. Prove that, with the above definition of solution vector (1.43), the momentum and Laplace equations for the incompressible case can be cast in the matrix form

$$\frac{d}{dr} \mathbf{y} = \mathbf{A} \cdot \mathbf{y}, \quad (1.50)$$

where the matrix $\mathbf{A}_l(r)$ reads

$$\begin{pmatrix} -\frac{2}{r} & \frac{l(l+1)}{r} & 0 & 0 & 0 & 0 \\ -\frac{1}{r} & \frac{1}{r} & 0 & \frac{1}{r} & 0 & 0 \\ \frac{4}{r} \left(\frac{3\mu}{r} - \rho_0 g \right) & -\frac{l(l+1)}{r} \left(\frac{6\mu}{r} - \rho_0 g \right) & 0 & \frac{l(l+1)}{r} & -\frac{\rho_0(l+1)}{r} & \rho_0 \\ -\frac{1}{r} \left(\frac{6\mu}{r} - \rho_0 g \right) & \frac{2(2l^2+2l-1)\mu}{r^2} & -\frac{1}{r} & -\frac{3}{r} & \frac{\rho_0}{r} & 0 \\ -4\pi G \rho_0 & 0 & 0 & 0 & -\frac{l+1}{r} & 1 \\ -\frac{4\pi G \rho_0(l+1)}{r} & \frac{4\pi G \rho_0 l(l+1)}{r} & 0 & 0 & 0 & \frac{l-1}{r} \end{pmatrix}, \quad (1.51)$$

with the gravity $g = 4\pi G \rho_0 r/3$.

Deriving equation (1.47) with respect to r and summing the result of this derivation to (1.47) multiplied by $2/r$ and to (1.48) multiplied by $-l(l+1)/r^2$, we obtain

$$\begin{aligned} \nabla_r^2(\rho_0 \phi_l - \rho_0 g U_l + \Pi_l) &= -2\mu \partial_r^3 U_l - \frac{8}{r^2} \mu \partial_r^2 U_l \\ &+ \frac{2l(l+1)}{r^2} \mu \partial_r U_l + \frac{4l(l+1)}{r^3} \mu U_l + \frac{2l(l+1)}{r^2} \mu \partial_r^2 V_l - \frac{2[l(l+1)]^2}{r^3} \mu V_l \end{aligned} \quad (1.52)$$

and finally, making use of equation (1.49) derived twice with respect to r in order to eliminate $\partial_r^3 U_l$, we obtain

$$\nabla_r^2(\rho_0 \phi_l - \rho_0 g U_l + \Pi_l) = 0, \quad (1.53)$$

where

$$\nabla_r^2 = \partial_r^2 + \frac{2}{r} \partial_r - \frac{l(l+1)}{r^2}, \quad (1.54)$$

which is equation (25) in Wu and Peltier (1982), multiplied by ρ_0 .

From (1.47) by collecting the derivative with respect to r ,

$$\begin{aligned} \partial_r(\rho_0 \phi_l - \rho_0 g_0 U_l + \Pi_l) &= -2\mu \partial_r^2 U_l \\ &- \frac{\mu}{r^2} [4r \partial_r U_l - 4U_l + l(l+1)(-U_l - r \partial_r V_l + 3V_l)], \end{aligned} \quad (1.55)$$

which can be rearranged as follows

$$\begin{aligned} \partial_r(\rho_0\phi_l - \rho_0g_0U_l + \Pi_l) = -2\mu\partial_r^2U_l \\ - \frac{\mu}{r^2}[4r\partial_rU_l - 4U_l - l(l+1)U_l - rl(l+1)\partial_rV_l + 3l(l+1)V_l], \end{aligned} \quad (1.56)$$

which becomes with equation (1.49)

$$\partial_r(\rho_0\phi_l - \rho_0g_0U_l + \Pi_l) = -\mu\partial_r^2U_l - 4\frac{\mu}{r}\partial_rU_l - 2\frac{\mu}{r^2}U_l + \frac{\mu}{r^2}l(l+1)U_l. \quad (1.57)$$

Multiplying equation (1.55) by r^2 and taking into account equation (1.57) yields, after having changed sign in each term,

$$r^2\partial_r(\rho_0gU_l - \Pi_l - \rho_0\phi_l) = \mu r^2\partial_r^2U_l + 4\mu r\partial_rU_l + 2\mu U_l - \mu l(l+1)U_l. \quad (1.58)$$

We define

$$\rho_0gU_l - \rho_0\phi_l - \Pi_l = \Gamma_l. \quad (1.59)$$

Exercise. Show that the solution of the Laplace equation (1.46) takes the form

$$\phi = c_3r^l + c_3^*r^{-(l+1)}, \quad (1.60)$$

where r^l denotes the regular solution in $r = 0$ and $r^{-(l+1)}$ denotes the singular one. The subscript 3 in equation (1.60) is used for convenience.

Γ_l satisfies the Laplace equation (1.53) and thus takes the following form, with the same dependence of ϕ_l with respect to r , where the constants c_1 and c_1^* , with subscript 1, are multiplied by μ for convenience, as will be apparent in the following

$$\Gamma_l = \mu c_1r^l + \mu c_1^*r^{-(l+1)}. \quad (1.61)$$

The homogeneous equation

$$r^2\partial_r^2U_l + 4r\partial_rU_l + 2U_l - l(l+1)U_l = 0, \quad (1.62)$$

obtained from equation (1.57) with the left member set to zero, has two solutions, a regular one

$$U_l' = c_2r^{(l-1)} \quad (1.63)$$

and a singular one

$$U_l'' = c_2^* r^{-(l+2)}. \quad (1.64)$$

A particular solution for the regular component can be obtained by substituting the regular component of Γ , providing

$$r^2 \partial_r^2 U_l + 4r \partial_r U_l + 2U_l - l(l+1)U_l = r^2 c_1 l r^{(l-1)}. \quad (1.65)$$

The regular solution is thus

$$U_l''' = \frac{c_1 l}{2(2l+3)} r^{(l+1)}. \quad (1.66)$$

The singular component of the solution becomes, with the same procedures, becomes

$$U_l'''' = c_1^* \frac{(l+1)}{2(2l-1)} r^{-l}. \quad (1.67)$$

Summing up all the contributions, we obtain

$$U_l = c_1 \frac{l r^{(l+1)}}{2(2l+3)} + c_2 r^{(l-1)} + c_1^* \frac{(l+1)}{2(2l-1)} r^{-l} + c_2^* r^{-(l+2)}. \quad (1.68)$$

From equation (1.68) and (1.49) we obtain the horizontal component of the displacement

$$V_l = c_1 \frac{l+3}{2(2l+3)(l+1)} r^{l+1} + c_2 \frac{r^{l-1}}{l} + c_1^* \frac{2-l}{2l(2l-1)} r^{-l} - c_2^* \frac{r^{-(l+2)}}{l+1}. \quad (1.69)$$

Exercise. Verify that with the definitions of the solution vector (1.43) the components y_3 , y_4 and y_6 take the form

$$\begin{aligned} y_3 = & c_1 \left[\frac{(l\rho_0 gr + 2(l^2-l-3)\mu)r^l}{2(2l+3)} \right] + c_2 [\rho_0 gr + 2(l-1)\mu] r^{l-2} \\ & + c_3 \left[-\rho_0 r^l \right] + c_1^* \left[\frac{(l+1)\rho_0 gr - 2(l^2+3l-1)\mu}{2(2l-1)r^{l+1}} \right] \\ & + c_2^* \left[\frac{\rho_0 gr - 2(l+2)\mu}{r^{l+3}} \right] + c_3^* \left[-\rho_0 r^{-(l+1)} \right] \end{aligned} \quad (1.70)$$

$$\begin{aligned} y_4 = & c_1 \frac{l(l+2)}{(2l+3)(l+1)} r^l + c_2 \frac{2(l-1)}{l} r^{(l-2)} \\ & + c_1^* \frac{(l^2-1)}{l(2l-1)} r^{-(l+1)} + c_2^* \frac{2(l+2)}{l+1} r^{-(l+3)} \end{aligned} \quad (1.71)$$

$$\begin{aligned}
 y_6 = & +c_1 \frac{2\pi G \rho l r^{l+1}}{(2l+3)} + c_2 4\pi G \rho_0 r^{l-1} - c_3 (2l+1) r^{l-1} \\
 & + c_1^* \frac{2\pi G(l+1)}{(2l-1)} r^{-l} + 4\pi G \rho_0 c_2^* r^{-(l+2)}.
 \end{aligned} \tag{1.72}$$

For each of the N layers of the earth model (assuming that each layer has material parameters which are constant inside it and that gravity g is constant inside such a layer), on the basis of equations (1.60), (1.68) - (1.72) the solution can thus be written as

$$\mathbf{y}(r) = \mathbf{Y}_l(r) \cdot \mathbf{C}_l, \tag{1.73}$$

in which $\mathbf{Y}_l$ is the fundamental matrix and $\mathbf{C}_l$ a 6-component vector integration constant.

The fundamental matrix $\mathbf{Y}_l(r, s)$ reads

$$\left(\begin{array}{ccc}
 \frac{l r^{l+1}}{2(2l+3)} & r^{l-1} & 0 \\
 \frac{(l+3)r^{l+1}}{2(2l+3)(l+1)} & \frac{r^{l-1}}{l} & 0 \\
 \frac{(l\rho_0 g r + 2(l^2 - l - 3)\mu)r^l}{2(2l+3)} & (\rho_0 g r + 2(l-1)\mu)r^{l-2} & -\rho_0 r^l \\
 \frac{l(l+2)\mu r^l}{(2l+3)(l+1)} & \frac{2(l-1)\mu r^{l-2}}{l} & 0 \\
 0 & 0 & -r^l \\
 \frac{2\pi G \rho_0 l r^{l+1}}{2l+3} & 4\pi G \rho_0 r^{l-1} & -(2l+1)r^{l-1}
 \end{array} \quad \dots \right)$$

$$\left(\begin{array}{ccc}
 \frac{(l+1)r^{-l}}{2(2l-1)} & r^{-l-2} & 0 \\
 \frac{(2-l)r^{-l}}{2l(2l-1)} & -\frac{r^{-l-2}}{l+1} & 0 \\
 \frac{(l+1)\rho_0 g r - 2(l^2 + 3l - 1)\mu}{2(2l-1)r^{l+1}} & \frac{\rho_0 g r - 2(l+2)\mu}{r^{l+3}} & -\frac{\rho_0}{r^{l+1}} \\
 \dots & \frac{(l^2 - 1)\mu}{l(2l-1)r^{l+1}} & \frac{2(l+2)\mu}{(l+1)r^{l+3}} \\
 0 & 0 & -\frac{1}{r^{l+1}} \\
 \frac{2\pi G \rho_0 (l+1)}{(2l-1)r^l} & \frac{4\pi G \rho_0}{r^{l+2}} & 0
 \end{array} \right) \tag{1.74}$$

Each column of this fundamental matrix represents an independent solution of the system (1.50) of ordinary differential equations. The analytical expression of the fundamental solution, which includes the regular and singular part in $r = 0$, was first obtained in Sabadini *et al.* (1982a), while the regular part, which is appropriate for the solution of a homogeneous, viscoelastic sphere, was first obtained by Wu and Peltier (1982). The inverse of the fundamental matrix $\mathbf{Y}_l$ has the form

$$\mathbf{Y}_l^{-1}(r) = \mathbf{D}_l(r) \bar{\mathbf{Y}}_l(r), \quad (1.75)$$

with $\mathbf{D}$ being a diagonal matrix with elements

$$\text{diag}(\mathbf{D}_l(r)) = \frac{1}{2l+1} \left(\frac{l+1}{r^{l+1}}, \frac{l(l+1)}{2(2l-1)r^{l-1}}, -\frac{1}{r^{l-1}}, lr^l, \frac{l(l+1)}{2(2l+3)}r^{l+2}, -r^{l+1} \right) \quad (1.76)$$

and

$$\bar{\mathbf{Y}}_l(r) = \begin{pmatrix} \frac{\rho g r}{\mu} - 2(l+2) & 2l(l+2) & -\frac{r}{\mu} & \frac{lr}{\mu} & \frac{\rho r}{\mu} & 0 \\ -\frac{\rho g r}{\mu} + \frac{2(l^2+3l-1)}{l+1} & -2(l^2-1) & \frac{r}{\mu} & \frac{(2-l)r}{\mu} & -\frac{\rho r}{\mu} & 0 \\ 4\pi G\rho & 0 & 0 & 0 & 0 & -1 \\ \frac{\rho g r}{\mu} + 2(l-1) & 2(l^2-1) & -\frac{r}{\mu} & -\frac{(l+1)r}{\mu} & \frac{\rho r}{\mu} & 0 \\ -\frac{\rho g r}{\mu} - \frac{2(l^2-l-3)}{l} & -2l(l+2) & \frac{r}{\mu} & \frac{(l+3)r}{\mu} & -\frac{\rho r}{\mu} & 0 \\ 4\pi G\rho r & 0 & 0 & 0 & 2l+1 & -r \end{pmatrix} \quad (1.77)$$

Although it would be quite laborious to derive such an analytical compact form of a 6×6 inverse matrix ‘by hand’, this can be done nowadays by means of an algebraic software package like *Mathematica*. It was first done by Spada *et al.* (1990, 1992b). Of course, it is not difficult to show analytically that $\mathbf{Y}_l \times \mathbf{Y}_l^{-1} = \mathbf{I}$, with $\mathbf{I}$ the identity matrix, by hand! Appendix A provides the elements of the fundamental matrix for the compressible case (Vermeersen *et al.*, 1996b).

3.2 TOROIDAL SOLUTION FOR THE INCOMPRESSIBLE CASE

With the following definition of the toroidal vector solution $\mathbf{y}$

$$y_1 = W_l \quad (1.78)$$

$$y_2 = \mu \left(\frac{dW_l}{dr} - \frac{W_l}{r} \right), \quad (1.79)$$

the toroidal differential equations take the following form for each harmonic degree l

$$\frac{d}{dr} \mathbf{y} = \mathbf{A} \cdot \mathbf{y}. \quad (1.80)$$

where the $\mathbf{A}$ matrix has been obtained by Alterman *et al.* (1959)

$$\mathbf{A}_l(r) = \begin{pmatrix} \frac{1}{r} & \frac{1}{r} \\ \frac{\mu(l(l+1)-2)}{r^2} & \frac{\mu}{-3r} \end{pmatrix}. \quad (1.81)$$

Exercise. Show that the l component of the fundamental solution for the toroidal solution is given by

$$\mathbf{Y}_l(r) = \begin{pmatrix} r^l & r^{-l-1} \\ \mu(l-1)r^{l-1} & -\mu(s)(l+2)r^{-l-2} \end{pmatrix}. \quad (1.82)$$

The inverse matrix of the fundamental toroidal solution reads

$$\mathbf{Y}_l^{-1}(r) = \begin{pmatrix} \frac{2+l}{r^l(1+2l)} & \frac{r^{1-l}}{\mu(1+2l)} \\ \frac{r^{1+l}(l-1)}{1+2l} & \frac{r^{2+l}}{-\mu(1+2l)} \end{pmatrix}. \quad (1.83)$$

4. FUNDAMENTAL SPHEROIDAL MATRIX IN THE LAPLACE DOMAIN

The Laplace transform $\tilde{f}(s)$ of a function $f(t)$ is defined by

$$\tilde{f}(s) = \int_0^\infty f(t)e^{-st}dt, \quad (1.84)$$

with t as time and s the Laplace variable (which has the dimension of inverse time). It is straightforward to show that the Laplace transform of the time derivative df/dt of the function $f(t)$ is $s\tilde{f}(s)$.

All the results shown in this book are based on the incompressible viscoelastic Maxwell solid, which means that only the rigidity μ enters the fundamental solution (see equation (1.74)). We can thus make use of the one-dimensional relationship between the stress and strain rate for a Maxwell solid depicted in Figure 1.2 as given by

$$\frac{d\epsilon}{dt} = \frac{\sigma}{2\nu} + \frac{1}{2\mu} \frac{d\sigma}{dt}, \quad (1.85)$$

with ν being the viscosity of the dashpot (of the mantle, for the case of the Earth!). Laplace transformation of the equation above leads to

$$\tilde{\sigma}_{ij}(s) = 2\tilde{\mu}(s)\tilde{\epsilon}_{ij}(s), \quad (1.86)$$

with the Laplace-transformed Lamé parameter $\mu(s)$ being

$$\tilde{\mu}(s) = \frac{\mu s}{s + \mu/\nu}. \quad (1.87)$$

Note that equation (1.86) has the form of a Hookean (linearly elastic) equation in the Laplace-transformed domain. This is a very important aspect and greatly facilitates calculations. So we can derive equations for linear Maxwell viscoelastic bodies in the time domain with formulas for linear Hooke elastic bodies in the Laplace-transformed domain. It can be shown that this is generally valid for all linear viscoelastic bodies (so also, e.g., the Kelvin-Voigt and Burgers models in Ranalli, 1995). The so-called Correspondence Principle states that by calculating the associated elastic solutions in the Laplace-transformed domain the time dependent viscoelastic solutions can be found by Laplace inversion in a unique way. From now on, in the other sections of this chapter, the tilde over the quantities will be neglected in order not to overwhelm the text, although all the equations and quantities are defined in the Laplace-transformed domain.

5. PROPAGATOR MATRIX TECHNIQUE

For each layer of a spherical earth model, the solution vector (1.43) can be determined from the fundamental matrix. This solution vector expresses the most general solution for displacements (radial and horizontal), stresses (radial and tangential), gravity and y_6 , from which the gravity gradient can be derived for each layer of the spherical model and for each harmonic degree l in the Laplace domain. Each viscoelastic layer of the model is bounded by either another internal viscoelastic layer or an external layer (free outer surface, inviscid outer core layer at the core-mantle boundary (CMB)). For each of these cases we need to determine the boundary conditions.

The internal boundary conditions are quite easy: for a boundary between two elastic or viscoelastic layers we require that U_l , V_l , σ_{rrl} , $\sigma_{r\theta l}$ and ϕ_l be continuous. This implies that during deformation there will be no ‘cavitation’ and no slip, while it is also assumed that no material crosses the boundary. Internal boundaries where no material crosses are called chemical boundaries. Internal boundaries where material does cross, undergoing a phase change, are called phase-change boundaries. The boundary between the upper mantle and lower mantle at about 670 km depth is likely to be partly a chemical and partly a phase-change boundary, but we will assume first that in our earth models there are only chemical boundaries. In the following section 1.7, phase-change boundaries will also be tackled.

As was already alluded to when y_6 was defined in equation (1.43), we do not take the gravity gradient as sixth component of the vector but a combination of gravity, gravity gradient and radial displacement. The reason becomes clear when the boundary condition for the gravity gradient at the free outer surface of the model is considered. If ϕ^e denotes the external potential and ϕ the potential evaluated in the top layer of the earth model, then applying the Gauss theorem at the Poisson equation within a volume embedded in a pill-box at the free

surface, with $\rho_1 = -\rho_0 \nabla \cdot \mathbf{u}$ from equation (1.13), the surface layer being homogeneous in density, we obtain

$$\frac{\partial \phi_l^e}{\partial r} - \frac{\partial \phi_l}{\partial r} = -4\pi G \rho U_l. \quad (1.88)$$

Only the radial component of the gravitational potential and the radial component of the displacement contribute to the surface integral.

The gravity gradient of the external layer resulting from the term in $r^{-(l+1)}$ from equation (1.60) satisfies

$$\frac{\partial \phi_l^e}{\partial r} = -\frac{l+1}{r} \phi_l^e. \quad (1.89)$$

The contribution from the other term from equation (1.60), proportional to r^l , becomes irregular at infinity and must be excluded. We also have

$$\phi_l^e = \phi_l, \quad (1.90)$$

so that we can express the external boundary condition as

$$y_6 = -\frac{\partial \phi}{\partial r} - \frac{l+1}{r} \phi + 4\pi G \rho U_l = 0. \quad (1.91)$$

With this it is clear that also y_6 is continuous for internal boundaries between viscoelastic layers.

5.1 PROPAGATION OF THE SPHEROIDAL SOLUTION

Due to the continuity of the fields y_j ($j = 1, 2, 3, 4, 5, 6$) at the interface $r = r_{i+1}$, the top layer i , in which

$$\mathbf{y}^{(i)}(r_{i+1}, s) = \mathbf{Y}^{(i)}(r_{i+1}, s) \mathbf{C}^{(i)}, \quad (1.92)$$

can be linked to the layer $i+1$ below it, with

$$\mathbf{y}^{(i+1)}(r_{i+1}, s) = \mathbf{Y}^{(i+1)}(r_{i+1}, s) \mathbf{C}^{(i+1)} \quad (1.93)$$

by assuming the continuity of the components of the solution vector

$$\mathbf{y}^{(i)} = \mathbf{y}^{(i+1)} \quad (1.94)$$

as a consequence of the boundary conditions at the internal boundaries. The subscript l denoting the harmonic degree is deleted from now on, in order to not overwhelm the notation. With equations (1.92)-(1.94) it is possible to express the unknown constant vector $\mathbf{C}^{(i)}$ in terms of the unknown constant vector $\mathbf{C}^{(i+1)}$. By doing this for every internal boundary of an N layer model (layer 1

is the top layer [crust or lithosphere], layers 2, 3, ..., $N - 1$ the layers below it, and layer N the core), the solution vector at the surface of the Earth at $r_1 = a$ can be related to the conditions $\mathbf{C}_l^{(N)}(r_c)$ at the core-mantle boundary (CMB) $r_N = r_c$ as

$$\mathbf{y}(a, s) = \left(\prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \right) \mathbf{Y}^{(N)}(r_c, s) \mathbf{C}^{(N)}. \quad (1.95)$$

The conditions at the CMB have been disputed among geophysicists since the 1960's. This controversy focuses on the treatment of the continuity conditions for the vertical deformation at the CMB. Without going into details, if it is required that the vertical deformation at the CMB be continuous, then this restricts the core to being either in a state of neutral equilibrium (homogeneous with neutral adiabatic temperature gradient) or that the radial stress at the CMB is zero. Both could be the case, but such restrictions are obviously not always the case in reality. Therefore the vertical deformation should in general not be continuous at the CMB. This might seem strange, as one would think that this could lead to 'cavitation' or to the overlapping of layers. The way out of this conundrum is that the fluid core layers are rather to be interpreted as equipotentials rather than material layers.

Gravity should be continuous at the CMB, at least if we assume that there are no additional masses positioned at the CMB. Inside the core, gravity should be proportional to r^l , as the other solution of equation (1.60) is irregular at the center of the Earth. So for the lowermost mantle layer at the CMB we get

$$y_5^{(N)}(r_c) = K_1 r_c^l, \quad (1.96)$$

with K_1 a constant.

Assuming that the core is inviscid (fluid), we can readily deduce that the tangential displacement of the mantle is not restricted, so for the lowermost mantle layer at the CMB we have

$$y_2^{(N)}(r_c) = K_2, \quad (1.97)$$

with K_2 as a constant and $y_2^{(N)}$ the second component of the vector $\mathbf{y}^{(N)}$.

The following condition holds for the lowermost mantle layer at the CMB, where the first term provides the radial displacement of the equipotential ϕ/g_c (note the minus sign of ϕ in (1.43))

$$y_1^{(N)}(r_c) = -\frac{y_5^{(N)}}{g_c} + K_3 = -\frac{3r_c^{l-1}}{4\pi G \rho_c} K_1 + K_3, \quad (1.98)$$

with g_c being the gravity at the CMB, K_3 a constant, ρ_c the density of the core and $y_1^{(N)}$ the first component of the vector $\mathbf{y}^{(N)}$.

The radial stress (pressure) should be continuous over the CMB. This leads for the lowermost mantle layer at the CMB to the condition

$$y_3^{(N)}(r_c) = g_c \rho_c K_3 = \frac{4}{3} \pi G \rho_c^2 r_c K_3, \quad (1.99)$$

with $y_3^{(N)}$ the third component of the vector $\mathbf{y}^{(N)}$.

The tangential stress in the fluid core is zero, and thus continuity of stress requires for the lowermost mantle layer at the CMB that

$$y_4^{(N)}(r_c) = 0, \quad (1.100)$$

with $y_4^{(N)}$ as the fourth component of the vector $\mathbf{y}^{(N)}$.

Finally, y_6 should also be continuous at the CMB, leading for the lowermost mantle layer at the CMB to the condition

$$y_6^{(N)}(r_c) = 2(l-1)r_c^{l-1}K_1 + 4\pi G \rho_c K_3, \quad (1.101)$$

with $y_6^{(N)}$ as the sixth component of the vector $\mathbf{y}^{(N)}$.

If we treat the core as the innermost layer, then the conditions at the CMB can be expressed as a 6×3 interface matrix $\mathbf{I}_c(r_c)$ as

$$\mathbf{Y}^{(N-1)}(r_c, s) \mathbf{C}^{(N-1)}(r_c) = \mathbf{I}_c(r_c) \mathbf{C}_c, \quad (1.102)$$

with

$$\mathbf{I}_c(r_c) = \begin{pmatrix} -r_c^{l-1}/A_c & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \rho_c A_c r_c \\ 0 & 0 & 0 \\ r_c^l & 0 & 0 \\ 2(l-1)r_c^{l-1} & 0 & 3A_c \end{pmatrix}, \quad (1.103)$$

with ρ_c being the (uniform) density of the core, $A_c = \frac{4}{3} \pi G \rho_c$, and $\mathbf{C}_c = (K_1, K_2, K_3)$ a 3-component constant vector.

The solution vector $\mathbf{y}(a, s)$, equation (1.95), with (1.103) can be used to express either the conditions for a free surface, or the conditions for a surface or internal loading. The loading/forcing case will be treated afterwards.

The solution vector $\mathbf{y}(a, s)$ can be split into two parts: one that contains the unconstrained parameters U_l , V_l and ϕ_l (which we are solving for), and the other containing the constrained $y_3 = \sigma_{rr}$, $y_4 = \sigma_{r\theta}$ and y_6 .

For a free surface, the components of the latter, as we have already seen, are all zero at the surface. If $\mathbf{P}_1$ denotes the projection operator on the third, fourth and sixth components of the solution vector given by equation (1.43), then we get the following condition

$$0 = \mathbf{P}_1 \mathbf{y}(a, s) = \mathbf{P}_1 \left(\prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \right) \mathbf{I}_c(r_c) \mathbf{C}_c, \quad (1.104)$$

with $\mathbf{P}_1$ the projection operator

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.105)$$

This condition constrains the s -values in the sense that only those s -values for which

$$\det \left(\mathbf{P}_1 \left(\prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \right) \mathbf{I}_c(r_c) \right) = 0 \quad (1.106)$$

are non-zero solutions of equation (1.104). The expression given by equation (1.106) is called the secular equation and the determinant the secular determinant. Its solutions $s = s_j$ ($j = 1, 2, 3, \dots, M$) are the inverse relaxation times of the M relaxation modes of the earth model. These s_j are dependent on the harmonic degree l (and thus must be determined for each harmonic degree), but the index l has been left out in order not to complicate the indexing. The total number of relaxation modes M for each harmonic degree is the same for each harmonic degree (with the exception of degree 1, but we will not digress any further on the differences between degree 1 on the one hand and degrees 2 and higher on the other).

Experience and (extremely laborious) analytical proofs have led to the following results:

- The surface contributes one mode, labeled $M0$.
- If there is an elastic lithosphere on top of a viscoelastic mantle, then there is one mode triggered by the lithosphere-mantle boundary, labeled $L0$.
- At the boundary of two viscoelastic layers, one buoyancy mode is triggered if the density on both sides of the boundary is different. Buoyancy modes between two mantle layers are usually labeled Mi , with $i = 1, 2, 3, \dots$, whereby $M1$ is usually the buoyancy mode associated with the 670 km discontinuity (upper / lower mantle) and $M2$ with the 400 km discontinuity (shallow upper mantle / mantle transition zone).

- At the same boundary two additional viscoelastic modes are triggered if the Maxwell time on both sides of the boundary is different (so if the viscosity and rigidity are different, but the ratio of viscosity and rigidity is not, then these viscoelastic modes are absent). These ‘paired’ modes are also called transient modes as they have relatively short relaxation times and are therefore usually labeled T_i , with $i = 1, 2, 3, \dots$
- The boundary between the lowermost mantle layer and the inviscid core contributes one mode, labeled $C0$.

It is thus possible, with the above rules, to determine the total number of modes of equation (1.106). This is of importance as solving this equation has to be done numerically. However, this root-solving is the only non-analytical part of the viscoelastic relaxation method as described in this chapter.

The root-solving procedure usually consists of two parts: grid-spacing, followed by a bisection algorithm. In the grid-spacing part, the s -domain is split into a number of discrete intervals. For each s -value at a boundary of an interval, the value of the determinant expressed by equation (1.106) is calculated, after which this value is multiplied with the value of the determinant of the s -value of the boundary next to it. If this product is positive, then the determinant has not changed in sign (or has changed an even number of times). If the product is negative, then we are sure that there is (at least) one root inside the interval bounded by the two s -values for which the determinant was calculated. In that case, the interval is split up in two parts, and the procedure of determining the product of the determinant of the bounding s -values is repeated. The interval where the determinant changes sign will result again in a negative product, and for this interval the procedure of cutting the interval in two, etc., is repeated. Thus the s -value where the determinant (1.106) is equal to zero becomes progressively better estimated with each further step in this bisection algorithm. Of course, it can happen that the determinant (1.106) changes sign over a small s -interval twice or even more times. It is thus necessary to choose small grids in the s -domain (in practice, it appears that especially the two modes of each T -mode pair have inverse relaxation times (s -values) that are very close to each other). Only after the complete number (determined with the rules above) of roots/modes of equation (1.106) has been found can one be sure that the complete signal will be retrieved after inverse-Laplace transformation. For this final step in the relaxation modeling procedure we use the so-called method of complex contour integration. Those readers who are not acquainted with this technique will find an overview in Appendix B.

5.2 PROPAGATION OF THE TOROIDAL SOLUTION

The same procedure discussed above can be used to propagate the toroidal solution. At the CMB the boundary condition is

$$y_2(c) = 0, \quad (1.107)$$

which states that at the core-mantle boundary the tangential stresses are zero (Smylie and Manshina, 1971).

Exercise. On analogy with the spheroidal case, it is possible to build $\mathbf{I}_c(r_c)$, which is now a vector, so as to make use of the same propagation procedure described for the spheroidal case. Making use of the boundary condition at the CMB for the tangential stress shows that $\mathbf{I}_c(r_c)$ takes the form

$$\mathbf{I}_c(r_c) = \begin{pmatrix} \frac{(l-1)r_c^{3l+1}}{l+2} + r_c^{-(l+1)} \\ 0 \end{pmatrix}. \quad (1.108)$$

6. INVERSE RELAXATION TIMES FOR INCOMPRESSIBLE EARTH MODELS

In order to gain insights into the physics of the relaxation processes, it is important to take a close look at the relaxation times corresponding to the modes excited by discontinuities in the physical parameters of simple earth models. We will consider the spheroidal case. The relaxation times for the five layer model, depicted in Figure 1.3, panel (a), are shown in Figures 1.4 and 1.5 as a function of the harmonic degree l . A simpler 4-layer model is shown in Figure 1.3, panel (b), for comparison.

The relaxation times $T_r = -1/s_i$ are expressed in years, ranging from $l = 2$ to $l = 100$. Figure 1.4 deals with a viscosity increase in the lower mantle, with the ratio B between the lower and upper mantle viscosity ranging from 1 to 200. OM stands for an old viscosity model in which the upper mantle viscosity is fixed at 10^{21} Pa s, while NM stands for a new viscosity model in which ν_1 is fixed at 0.5×10^{21} Pa s, in agreement with the recent analyses by Lambeck *et al.* (1990), Vermeersen *et al.* (1999) and Devoti *et al.* (2001) based on postglacial rebound modeling from different perspectives, sea-level changes in the far field and long-wavelength geopotential variations. These models are chemically stratified at 420 and 670 km depth, Figure 1.3 panel (a), and the viscosity is uniform in the whole upper mantle; this stratification supports nine relaxation modes. The slowest modes have been named $M1$ and $M2$ by Wu and Peltier (1982) and are associated with the two internal chemical boundaries at 670 and 420 km. These $M1$ and $M2$ modes will be quoted several times in the book when dealing with the geophysical processes affected by the slow readjustment of the 670 and 420 km density discontinuities.

At low degrees, Figure 1.4, they are followed by the lithospheric ($L0$) mode and by the core ($C0$) and mantle ($M0$) modes, as portrayed in the panel NM by $B = 1$, with $B = \nu_2/\nu_1$ denoting the ratio between the lower to upper

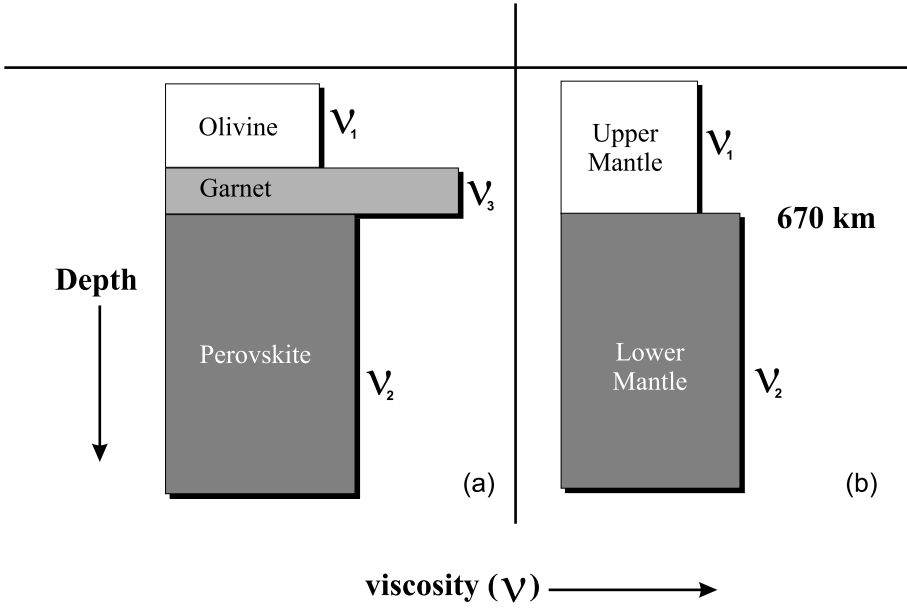


Figure 1.3. Schematic diagram with the rheological models, which include a hard transition zone, panel (a), considered for the evaluation of the relaxation times and a two-layer mantle, panel (b), shown for comparison.

mantle viscosity. When B is increased from 1 to 200, all the curves are moved upward toward slower relaxation times. This upward migration occurs first for longer wavelengths, say lower than $l = 10$, followed by the shorter ones, which are less affected by lower mantle viscosity. For shorter wavelengths only the $M1$, $M2$ and core modes have slower relaxation times, while the lithospheric and mantle modes are rather unaffected, the deformation at such high harmonic degrees being concentrated in the upper mantle and thus unaffected by lower mantle viscosity variations. The NM curves, in the left panel, can be obtained from their counterparts in the right panel by a uniform downward shift towards faster relaxation times, in agreement with the lowering of the global mantle viscosity of this model.

Figure 1.5 shows the effects of a viscosity increase in the transition zone for the new model NM, with $C = \nu_3/\nu_1$ denoting the ratio between the viscosity in the transition zone ν_3 with respect to the viscosity in the upper mantle. These models, with a stiff transition zone at the upper lower mantle boundary, are based on the laboratory studies by Karato (1989) and Meade and Jeanloz (1990), which suggest that the transition zone may form a layer of relatively high viscosity between the upper and lower mantle. The C parameter is varied between 1 and

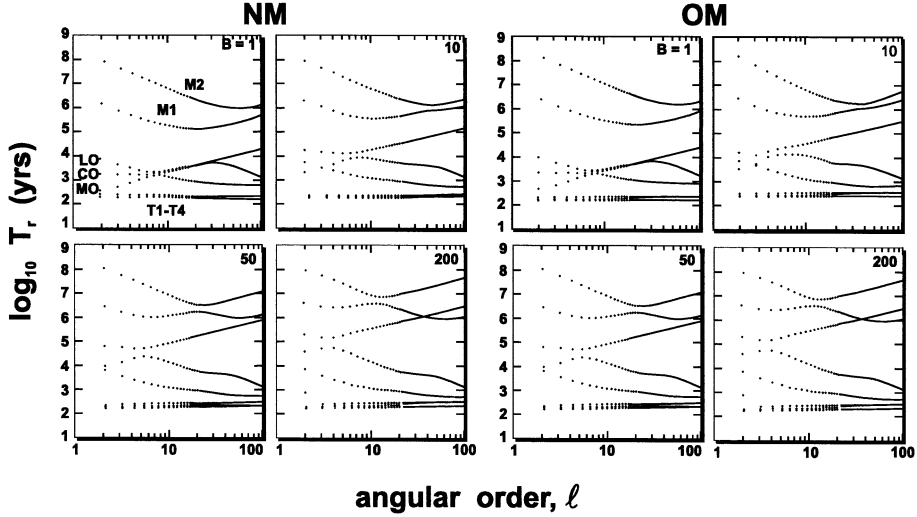


Figure 1.4. Relaxation times in years as a function of the harmonic degree l and varying lower mantle viscosity. The parameter $B = \nu_2/\nu_1$ is varied from 1 to 200. OM corresponds to $\nu_1 = 10^{21}$ Pa s, while NM corresponds to $\nu_1 = 0.5 \times 10^{21}$ Pa s. (Figure 2 in Spada *et al.*, 1992b).

200. Panel LB, with LB standing for lower bound, corresponds to an upper mantle viscosity of 0.5×10^{21} Pa s and the low value of $\nu_2 = 2 \times 10^{21}$ Pa s in the lower mantle, while UB, with UB standing for upper bound, corresponds to the same upper mantle viscosity and to a higher lower mantle viscosity of $\nu_2 = 2 \times 10^{22}$ Pa s. LB and UB for the lower mantle viscosity stand for the two possible viscosity solutions when true polar wander data and variations in the long-wavelength gravity field are used to constrain the viscosity of the lower mantle (see Chapters 4 and 5). Viscosity increase in the hard layer influences all the modes for all the models, in particular the $M1$ and $M2$ modes, which is not surprising as these modes are excited by the discontinuities that bound the region where the viscosity is varied. With respect to the previous figure, all the modes are now affected by the viscosity increase in the transition layer which, lying close to the surface, is also able to affect the short wavelength, high-degree modes.

7. PHASE-CHANGE INTERFACE

Density contrasts associated with phase changes that allow for material crossing the interface have been developed by Johnston *et al.* (1997) and applied to simplified 5-layer models. If the transition zone of the mantle behaves as a phase change, Johnston *et al.* (1997) have shown that the appropriate boundary

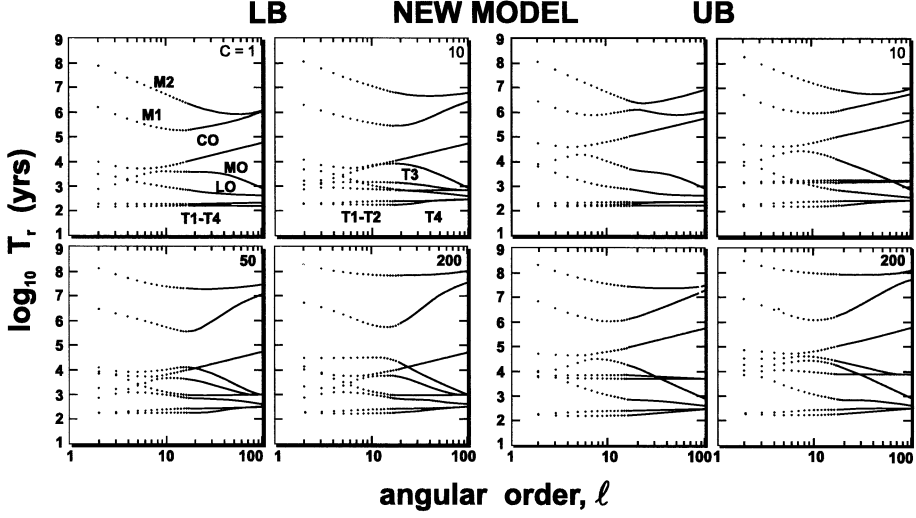


Figure 1.5. Relaxation times in years as a function of the harmonic degree l and varying lower mantle viscosity. The parameter $C = \nu_3/\nu_1$ is varied from 1 to 200. LB corresponds to $\nu_1 = 0.5 \times 10^{21}$ Pa s and $\nu_2 = 2 \times 10^{21}$ Pa s, while UB corresponds to $\nu_1 = 0.5 \times 10^{21}$ Pa s and $\nu_2 = 2 \times 10^{22}$ Pa s. (Figure 3 in Spada *et al.*, 1992b).

condition, with $i, i + 1$ denoting the two layers that match at the phase-change interface, is given by

$$\mathbf{Y}^{(i)}(r_{i+1}, s) \mathbf{C}^{(i)} = (\mathbf{I} + \xi_{i+1} \boldsymbol{\alpha}_{i+1} \boldsymbol{\beta}_{i+1}) \mathbf{Y}^{(i+1)}(r_{i+1}, s) \mathbf{C}^{(i+1)}, \quad (1.109)$$

where ξ_i denotes the response function for the i -th layer defined by Christensen (1985), $\mathbf{I}$ denotes the identity matrix and $\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i$ are given by

$$\boldsymbol{\alpha}_i = (\Delta\rho_i/\rho_{i-1}, 0, 0, 0, 0, 0) \quad (1.110)$$

$$\boldsymbol{\beta}_i = (-4\mu_i/r_i, 2l(l+1)\mu_i/r_i, -1, 0, 0, 0)/(\rho_i g(r_i)), \quad (1.111)$$

where $\Delta\rho_i = \rho_{i-1} - \rho_i$ denotes the density contrast. It should be noted that the definition of $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ differ from that given in the equation (30) by Johnston *et al.* (1997) since their procedure has been adapted to ours, where the inner layers correspond to increasing values of the index i . Equation (1.109) does not apply at rheological interfaces, like the lithosphere-mantle or core-mantle interface, where $\xi = 0$ is used.

In our propagation method, the secular equation (1.106) now becomes

$$\det \left(\mathbf{P}_1 \left(\prod_{i=1}^{N-1} (\mathbf{I} + \xi_{i+1} \boldsymbol{\alpha}_{i+1} \boldsymbol{\beta}_{i+1}) \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \right) \mathbf{I}_c(r_c) \right) = 0. \quad (1.112)$$

For an univariant phase change, the response factor ξ_i takes the following explicit expression as a function of various thermodynamic parameters

$$\xi_i = 1 - \frac{\rho_i^3 g k c_p}{T w (\rho_{i+1} - \rho_i) (dP_c/dT)^2}, \quad (1.113)$$

where k is the thermal diffusivity, c_p the specific heat at constant pressure, T the absolute temperature, w the vertical component of the background convection velocity and dP_c/dT is the Clapeyron slope, which gives the rate of change of the pressure of the phase change with respect to the temperature.

Once the parameters in Table 2 of Johnston *et al.* (1997) are used, it is found that an appropriate value for the response factor at 420 and 670 km is $\xi_i = 0.7$. The physical meaning of the response factor can be easily understood if we recall here its definition in terms of the difference between the displacement ε of the interface where the phase change occurs and the particle displacement u (Johnston *et al.*, 1997).

$$\varepsilon - u(r_0) = \xi \frac{p^\delta}{\rho_{i+1} g}, \quad (1.114)$$

where p^δ is the incremental pressure due to the surface loads. If $\xi = 0$ the density contrast moves with the material particles and it denotes a chemical interface, while for $\xi = 1$ the density discontinuity due to the phase change occurs at constant pressure and is thus termed isobaric; in this case, the density jump is totally adiabatic because the temperature does not play any role in controlling the final position of the interface. Intermediate values of the response factor, in the interval between $\xi = 0$ and $\xi = 1$, are responsible for a behavior of the phase change interface between those two end members.

For an upper mantle viscosity ν_1 fixed at 10^{21} Pa s, Figure 1.6 portrays the inverse relaxation times $-1/s_i$ for variations in the lower mantle viscosity ν_2 . The s_i are the zeros of the secular equations (1.106) and (1.112), top and bottom panels respectively, for the 5-layer model described in Table 2.2 where the densities are volume averages of the Preliminary Reference Earth Model (PREM) of Dziewonski and Anderson (1981).

The top panel corresponds to a chemically stratified mantle, while the bottom treats the case of a phase change with $\xi = 0.7$ at the 420 and 670 km interfaces. We notice, comparing the two panels of Figure 1.6, that the modification in the ξ parameters from 0 to 0.7 affects only the slowest buoyancy relaxation modes, as expected, with those corresponding to the phase change being slower. It is

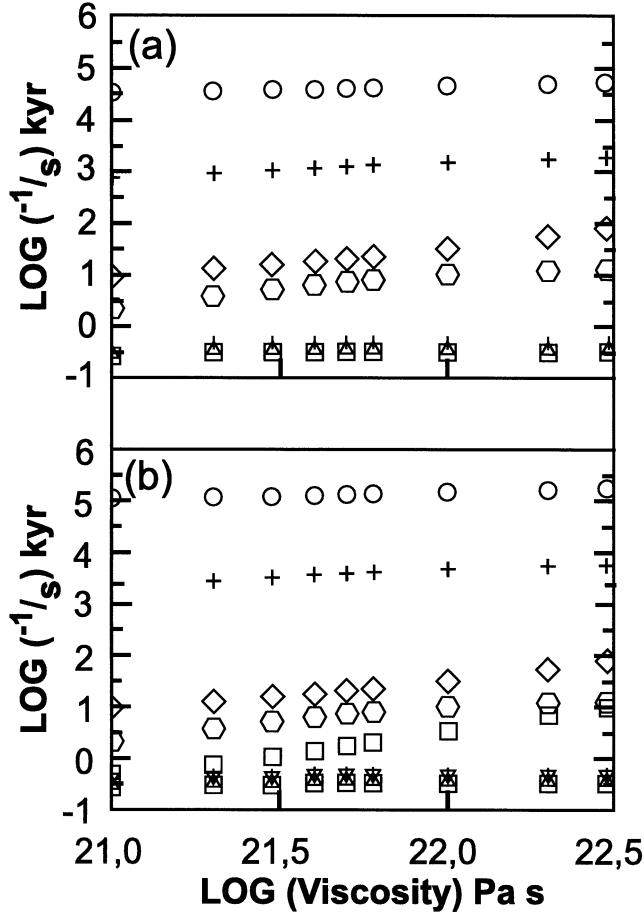


Figure 1.6. Inverse relaxation times for the isostatic modes, equations (1.106) and (1.112), for the 5-layer model of Table 2.2, for chemical and phase-change boundary conditions at 420 and 670 km, (a) and (b) panels respectively. The phase change is characterized by $\xi = 0.7$. The circles and the crosses correspond to the 420 and 670 km discontinuities. Redrawn from Figure 2 in Sabadini *et al.* (2002).

also interesting to note that these particular buoyancy modes are less sensitive to viscosity variation in the analyzed range than to changes in ξ , suggesting that the nature of the transition zone, whether chemical or phase change, must play a crucial role in the rotation dynamics of the Earth.

8. LOADING THE EARTH

The general solution of the following non-homogeneous system of ordinary differential equations for each harmonic degree l , where $\mathbf{f}$ is the vector charac-

terizing a general kind of source loading the Earth embedded at a depth r_s and $\mathbf{A}$ being the spheroidal or toroidal system matrix

$$\frac{d}{dr}\mathbf{y} = \mathbf{A} \cdot \mathbf{y} + \mathbf{f}, \quad (1.115)$$

is given by

$$\mathbf{y}(r) = \mathbf{Y}(r) \left[\int_{r_0}^r \mathbf{Y}^{-1}(r') \mathbf{f}(r') dr' + \mathbf{Y}^{-1}(r_0) \mathbf{y}(r_0) \right], \quad (1.116)$$

where r_0 denotes an interface lying below the source and $\mathbf{Y}(r)$ is the fundamental matrix given by equation (1.74) for the spheroidal case or equation (1.82) for the toroidal case; the harmonic degree l is not explicitly given.

In the following derivation it is assumed that the source is embedded in the outermost layer of radius a , denoting the radius of the Earth, and internal radius b , denoting the interface between the bottom of the lithosphere and underlying layer. This procedure can be generalized to a source embedded in an arbitrary internal layer. If the vector $\mathbf{f}$ has this form

$$\mathbf{f} = \mathbf{f} \delta(r - r_s), \quad (1.117)$$

with r_s denoting the radius of the source, the solution of the non-homogeneous system of ordinary differential equations takes the following form

$$\mathbf{y}(r) = \begin{cases} \mathbf{Y}(r) [\mathbf{Y}^{-1}(r_s) \mathbf{I} \mathbf{f} + \mathbf{Y}^{-1}(b) \mathbf{y}(b)], & r_s \leq r \leq a; \\ \mathbf{Y}(r) \mathbf{Y}^{-1}(b) \mathbf{y}(b), & b \leq r \leq r_s. \end{cases} \quad (1.118)$$

Exercise. Show that, if the forcing vector has the form

$$\mathbf{f}^* = \mathbf{f} \delta(r - r_s) + \mathbf{f}' \delta'(r - r_s), \quad (1.119)$$

the solution is given by

$$\mathbf{y}(r) = \begin{cases} \mathbf{Y}(r) [\mathbf{Y}^{-1}(r_s) (\mathbf{I} \mathbf{f} + \mathbf{A}(r_s) \mathbf{f}') + \mathbf{Y}^{-1}(b) \mathbf{y}(b)] & r_s \leq r \leq a; \\ \mathbf{Y}(r) \mathbf{Y}^{-1}(b) \mathbf{y}(b), & b \leq r \leq r_s, \end{cases} \quad (1.120)$$

where $\mathbf{A}$ is any of the matrices given by equation (1.51) or (1.81).

8.1 INTERNAL LOADING: EARTHQUAKES, SUBDUCTION

The boundary conditions for an internal forcing (earthquakes or internal density anomalies) are the vanishing of the stress components and y_6 at the Earth's surface:

$$y_3(a) = y_4(a) = y_6(a) = 0. \quad (1.121)$$

For internal density contrasts appropriate for modeling subduction processes, the $\mathbf{f}$ vector takes the form

$$\mathbf{f} = (0, 0, -\frac{1}{4\pi}g(r_s)(2l+1)/r_s^2, 0, 0, -G(2l+1)/r_s^2). \quad (1.122)$$

These conditions can be cast in the following form once we make use of equations (1.119) and (1.120)

$$\mathbf{P}_1 \mathbf{Y}(a)[\mathbf{Y}^{-1}(r_s)(\mathbf{I}\mathbf{f} + \mathbf{A}(r_s)\mathbf{f}') + \mathbf{Y}^{-1}(b)\mathbf{y}(b)] = 0, \quad (1.123)$$

where $\mathbf{P}_1$ denotes the projection operator on the third, fourth and sixth components of the solution vector defined in equation (1.105).

If the three-component vector $\mathbf{b}$ is defined in the following way

$$\mathbf{b} = -\mathbf{P}_1 \mathbf{Y}(a)\mathbf{Y}^{-1}(r_s)(\mathbf{I}\mathbf{f} + \mathbf{A}(r_s)\mathbf{f}'), \quad (1.124)$$

the boundary conditions at the surface become

$$\mathbf{P}_1 \mathbf{Y}(a)\mathbf{Y}^{-1}(b)\mathbf{y}(b) = \mathbf{b}, \quad (1.125)$$

which coincides with equation (52) in Sabadini *et al.* (1982a).

For earthquake sources, the appropriate $\mathbf{f}$ and $\mathbf{f}'$ vectors entering equation (1.120) can be found in Appendix C.

8.2 SURFACE LOADING: POINT MASS AND TIDAL FORCING

In the case of surface loading, the loading vector $\mathbf{f}$ affects only the stress components of the solution vector and the component related to the gradient of the perturbed potential y_3, y_4, y_6 . In the case of a point mass load acting at the Earth's surface, the $\mathbf{b}$ vector takes the form (Sabadini *et al.*, 1982a)

$$\mathbf{b} = (-\frac{1}{4\pi}g(a)(2l+1)/a^2, 0, -G(2l+1)/a^2). \quad (1.126)$$

which coincides with equation (1.122) once $r_s = a$.

For tidal forcing appropriate for solving problems related to changes in the centrifugal potential, as in the case of the Earth's rotation instabilities, the 3-component vector $\mathbf{b}$ becomes (Takeuchi *et al.*, 1962)

$$\mathbf{b} = (0, 0, -(2l+1)/a). \quad (1.127)$$

With these definitions, the boundary conditions at the surface for internal loading or surface loading become formally equivalent.

8.3 SOLVING FOR THE DISPLACEMENT AND PERTURBATION IN THE GRAVITATIONAL POTENTIAL

With the above definition for the $\mathbf{b}$ vector, whose expression depends on the forcing as shown previously, the boundary conditions at the Earth's surface become

$$\mathbf{b} = \mathbf{P}_1 \mathbf{y}(a, s) = \mathbf{P}_1 \left(\prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \right) \mathbf{I}_c(r_c) \mathbf{C}_c. \quad (1.128)$$

The unconstrained parameters U_l , V_l and ϕ_l at the surface can be expressed as

$$\begin{aligned} (U_l, V_l, -\phi_l)(a, s) &= \mathbf{P}_2 \mathbf{y}(a, s) \\ &= \left(\mathbf{P}_2 \prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \mathbf{I}_c(r_c) \right) \mathbf{C}_c, \end{aligned} \quad (1.129)$$

with $\mathbf{P}_2$ as the projection operator on the first, second and fifth component of the solution vector given by

$$\mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.130)$$

Elimination of $\mathbf{C}_c$ from equation (1.128) and (1.129) results in

$$\begin{aligned} (U_l, V_l, -\phi_l)(a, s) &= \left(\mathbf{P}_2 \prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \mathbf{I}_c(r_c) \right) \cdot \\ &\cdot \left(\mathbf{P}_1 \prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \mathbf{I}_c(r_c) \right)^{-1} \cdot \mathbf{b}, \end{aligned} \quad (1.131)$$

where $\mathbf{b}$ is any of the vectors defined in equation (1.124), (1.126) or (1.127).

Each of the M solutions s_j of equation (1.106) or (1.112) represents a singularity for the right hand member of equation (1.131). In fact, the quantity in the second brackets in the right member with

$$\mathbf{B} = \prod_{i=1}^{N-1} \mathbf{Y}^{(i)}(r_i, s) \mathbf{Y}^{(i)-1}(r_{i+1}, s) \quad (1.132)$$

can be written as

$$(\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c))^{-1} = (\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c))^{\dagger} / \det(\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c)), \quad (1.133)$$

where $(\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c))^{\dagger}$ denotes the matrix of the complementary minors. The determinant $\det(\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c))$ entering equation (1.133) can be written as $\prod_{i=1}^M (s - s_i)$, where the s_i are the solutions of equation (1.106) or (1.112): each s_i is thus responsible for the appearance of a singularity that corresponds to a first-order pole.

The inverse Laplace transform of equation (1.131) from the s to the time domain can thus be carried out by means of the residue theorem in Appendix B.3, as shown hereafter.

The inverse Laplace transform $f(t)$ of a function $\tilde{f}(s)$ is formally defined by complex contour integration by

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{f}(s) e^{st} ds, \quad (1.134)$$

in which the real constant γ is chosen such that singularities of $\tilde{f}(s)e^{st}$ are either all on the left or all on the right side of the vertical line running from $\gamma - i\infty$ to $\gamma + i\infty$. Closing the contour with a half-circle (either on the left of the line or on the right, depending on where the singularities are situated) leads to a complex contour that is known as the Bromwich path.

The residue theorem states that if $\tilde{f}(s)$ is an analytical function with M singularities of first order, in our case the right hand side of equation (1.131), then

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{f}(s) e^{st} ds = \sum_{i=1}^M [\text{Res}(\tilde{f}(s) e^{st})]_{s=s_j}, \quad (1.135)$$

where the residue in the pole of first order $s = s_j$ is given by (equation (B.30))

$$\text{Res}(\tilde{f}(s) e^{st})|_{s=s_j} = \lim_{s \rightarrow s_j} (s - s_j) \tilde{f}(s) e^{st}. \quad (1.136)$$

The above equation (1.136), with $(s - s_j)$ multiplying the function $\tilde{f}(s)$ in the s -domain and the exponential, tells us that, since at the denominator of equation (1.131) we have the product $\prod_{i=1}^M (s - s_i)$, we finally remain with $\prod_{i \neq j}^M (s_j - s_i)$ at the denominator after we have made use of it. It can be easily seen that $\prod_{i \neq j}^M (s_j - s_i)$ equals $\frac{d}{ds} \det(\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c))|_{s=s_j}$.

The solution of the field U_l , V_l and $-\phi_l$ at the surface of the Earth can thus be cast in the following form of the 3-component $\mathbf{K}(t)$ vector

$$(U_l, V_l, -\phi_l)(a, t) = \mathbf{K}_l(t) = \mathbf{K}_{le}(a)\delta(t) + \sum_{j=1}^M \mathbf{K}_{lj}(a)e^{s_j t}, \quad (1.137)$$

in which, on the basis of the equations above, the $\mathbf{K}_{lj}(a)$ are the vector residues of the solution kernel vector $\mathbf{y}(r, s)$ given by

$$\mathbf{K}_{lj}(a) = \left(\frac{\mathbf{P}_2 \mathbf{B} \mathbf{I}_c(r_c) \cdot (\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c))^\dagger}{\frac{d}{ds} \det(\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c))} \right)_{s=s_j} \cdot \mathbf{b}, \quad (1.138)$$

and $\mathbf{K}_{le}(a)$ denoting the elastic limit obtained by

$$\mathbf{K}_{le}(a) = \lim_{s \rightarrow \infty} (\mathbf{P}_2 \mathbf{B} \mathbf{I}_c(r_c) \cdot (\mathbf{P}_1 \mathbf{B} \mathbf{I}_c(r_c))^{-1}) \cdot \mathbf{b}. \quad (1.139)$$

For surface loading the dimensionless form of the first two components of the $\mathbf{K}_l$ vector are usually indicated by the Love numbers h_l, l_l for the radial and tangential displacement. The nondimensionalization constant is $\phi_{2,l}/g = a/M_E$, where $\phi_{2,l}$ is the potential perturbation due to the load and a and M_E denote the radius and the mass of the Earth. The third component is $\phi_{2,l}(1+k_l)$, where k_l is the Love number of the geopotential perturbation and 1 denotes the direct effect of the load.

The appearance of the delta $\delta(t)$ in the time domain is a consequence of the fact that equation (1.139) is a constant in the s -domain. These results provide the radially dependent part of the Green functions for the variables for each degree l .

Multiplying the Green functions with the Laplace-transformed forcing functions (which is the same as a convolution in the space-time domain) and performing an inverse Laplace transformation gives the sought-for expressions for the displacement fields and perturbation in the gravitational potential due to any kind of time-dependent loading.

Solution (1.137) shows that for each harmonic degree l the horizontal displacement, vertical displacement and change in gravity consist of an immediate response to the load (the elastic response), followed by M exponentially decaying (viscous) responses. At the least, the viscous responses are decaying only if the inverse relaxation times s_j for each harmonic degree are negative. For incompressible models this turns out to be always the case if the Earth layers show no density inversions in the radial Earth profile. However, if there is a layer with a greater density than its neighboring layer below, then the buoyancy mode for the interface will have a positive inverse relaxation time for each harmonic degree l . Such a positive relaxation time leads, according to equation (1.137), to an exponentially increasing response in the displacements and gravity variations, and thus the interface becomes Rayleigh-Taylor unstable. If this occurs, convective motions will be triggered in the earth model, and the

linearization assumed in the normal-mode theory as developed in this chapter breaks down.

From Chapter 3 onwards, we will make use of the superscript L in the Love numbers in order to denote a load, such as that associated with surface density anomalies, while the Love numbers corresponding to tidal loading will not carry any superscript.

9. APPROXIMATION METHOD FOR HIGH-DEGREE HARMONICS

When using spherical harmonics to describe Earth surface deformations, we always have to face the problem of how many terms we should sum up in order to obtain an accurate solution. Since every harmonic represents a standing wave on the Earth's surface, whose equator is about 40,000 km long, it is easy to determine the resolution given by each term of that series wherein the wavelength is given by the length of the equator divided by the harmonic degree. For point-like seismic sources, we find in the modeling carried out in Chapter 8 that this wavelength is uniquely related to the source depth for the elastic response and to the thickness of the elastic layer for (viscoelastic) relaxation: the summation of several thousand harmonics is thus required to get saturated convergence of the solution in the case of shallow earthquakes.

Modeling of the post-seismic effects of shallow and moderate-size earthquakes in the Mediterranean has motivated the development of new algorithms, with respect to those used for studying global post-seismic deformation as in Sabadini *et al.* (1984a) or Piersanti *et al.* (1995), appropriate for normal mode techniques at very high harmonic degrees, necessary for resolving wavelengths of a few hundred meters. These new algorithms are built on the multilayer code first developed in Vermeersen and Sabadini (1997), in which the core of the normal mode technique, namely the rootfinding procedures for multilayer models, had been modified and improved with respect to the old models (Sabadini *et al.*, 1984a; Piersanti *et al.*, 1995) in which the standard routines of rootfinding can deal with a five-layer model at the most.

The analytical propagator matrix technique, due to the stiffness of the fundamental matrices, does not allow, in practice, a straightforward calculation of more than a few thousands degrees. This is due to the $r^{\pm l}$ dependence of the fundamental matrix, equation (1.74), which causes numerical problems of over- and under-flow for high order harmonic degrees. The particular structure of the analytical fundamental solutions used in normal mode techniques thus does not allow a straightforward calculation, since numerical problems can readily occur due to the stiffness of the matrices used in the propagation routines. However, it is possible to mathematically demonstrate that the irregular fundamental solutions in non-homogeneous earth models are not necessary for calculating all the harmonic degrees, their weight getting smaller and smaller with increasing

degree. From a certain degree onwards, namely $l > 10^2 - 10^3$, depending on the earth model, it is possible to obtain an approximated expression of the fundamental solutions by keeping only the regular part. This makes it possible to remove the r^l growth of this part by rescaling procedures, as shown for the first time in Riva and Vermeersen (2002).

This section deals with a way of removing this stiffness problem by approximating the fundamental matrix solutions, followed by a rescaling procedure; in this way we can virtually go up to whatever harmonic degree is required. One way to cope with the stiffness problem is to use minors of the fundamental matrices (e.g., Woodhouse, 1988). This technique is commonly used in seismic applications that employ numerical integration by means of propagator matrices. Here we present a new alternative approximation technique that can be used for analytical propagator matrix models.

Here we will first demonstrate, by means of matrix algebra, that it is possible to obtain a first order approximation of the fundamental solutions by neglecting the irregular part of the fundamental matrix from a certain degree onwards. We will then determine the accuracy of this approximation and describe how to apply a reliable rescaling procedure. The following section is taken from Riva and Vermeersen (2002) and adapted to our formalism.

9.1 APPROXIMATION OF THE SOLUTION

Equation (1.106) for the inverse relaxation times s_j of the modes j can be changed into the following expression

$$\det[\mathbf{M}(r_1)\mathbf{B}'\mathbf{I}_c] = 0, \quad (1.140)$$

in which $\mathbf{M} = \mathbf{P}_1 \mathbf{Y}^{(1)}(r_1)$ is the fundamental matrix $\mathbf{Y}$ with the first, second and fifth rows deleted (i.e., the rows corresponding to the displacements and to the gravitational potential), $\mathbf{I}_c$ is a 6×3 matrix containing the conditions at the core-mantle boundary given by equation (1.103) and $\mathbf{B}'$ differs from equation (1.132) and is given by

$$\mathbf{B}' = \mathbf{Y}^{(1)-1}(r_2) \prod_{i=2}^{N-1} \mathbf{Y}^{(i)}(r_i) \mathbf{Y}^{(i)-1}(r_{i+1}). \quad (1.141)$$

With these definitions, the residues, i.e. the amplitudes of the modes j , for each of the roots s_j are thus given by

$$\mathbf{K}_{lj} = \left(\frac{\mathbf{N}(r_1)\mathbf{B}'\mathbf{I}_c \cdot [\mathbf{M}(r_1)\mathbf{B}'\mathbf{I}_c]^\dagger}{\frac{d}{ds} \det[\mathbf{M}(r_1)\mathbf{B}'\mathbf{I}_c]} \right)_{s=s_j} \cdot \mathbf{b}, \quad (1.142)$$

where the vector $\mathbf{b}$ is defined in section 1.8.2 for surface loading, $\mathbf{N}$ is $\mathbf{Y}^{(1)}(r_1)$ with the third, fourth and sixth rows deleted (corresponding to the stresses and the potential gradient) and the $\dagger$ symbol denotes the $(-1)^{i+j}$ -transposed of the nine minor determinants of the matrix between the brackets, with i and j denoting the number of rows and columns. The elastic response is given as the limit for s going to infinity of the same relation (1.142), without the s -derivative.

We now focus on the incompressible case; explicit expressions for $\mathbf{Y}$ for the compressible case can be found in Appendix A.

The problem we have to deal with in calculating residuals given by equation (1.142) is that the fundamental matrix $\mathbf{Y}$ contains three regular functions r^l and three irregular r^{-l} . The radii are non-dimensionalized by division by the average Earth radius a and are thus represented by a number between zero and one; it is obvious that

$$\lim_{l \rightarrow \infty} r^l \rightarrow 0 \quad \text{and} \quad \lim_{l \rightarrow \infty} r^{-l} \rightarrow +\infty. \quad (1.143)$$

Especially for deep layers ($r \simeq 1/2$ at the CMB), and due to the stiffness of the matrices, numerical evaluation of the fundamental solutions can represent a serious problem for high-degree harmonics; for practical purposes, then, we need to find a way to rescale equations (1.140) and (1.142). However, the propagation matrix technique, which is at the basis of the explicit form for $\mathbf{B}'$ (or $\mathbf{B}$ in equation (1.132)), does not allow a simple rescaling due to the non-commutativity of the matrix algebra. The fundamental matrix is stiff and this stiffness, in turn, would require rescaling by means of a transformation like $\mathbf{Y}' = \mathbf{K}\mathbf{Y}$, with $\mathbf{K}$ being a 6×6 linear operator. But once we put all the $\mathbf{Y}'$ and $(\mathbf{Y}')^{-1}$ into (1.140) and (1.142) we have no way, at the end of the calculations, to find an inverse transformation which can produce the actual values for the relaxation times and their residuals.

The explicit analytical form of the layer propagator matrix $\mathbf{Y}\mathbf{Y}^{-1}$ can be found in Martinec and Wolf (1998); note, however, that use of the explicit analytical form of the propagator matrix in the propagator matrix technique does not solve the stiffness problem.

However, there is an approximation method that does solve the stiffness problem, as discussed in the following section.

The basic idea underlying the approximation procedure is to consider the 6×6 matrix $\mathbf{Y}$ as formed by two 6×3 matrices, one regular ($\mathbf{Y}_R$) and one irregular ($\mathbf{Y}_I$),

$$\mathbf{Y} = [\mathbf{Y}_R \quad \mathbf{Y}_I]. \quad (1.144)$$

This splitting can also be found in Sabadini *et al.* (1982a). The first three rows of the inverse $\mathbf{Y}^{-1}$ can be cast as a 3×6 irregular matrix that we define as $\mathbf{Y}_U^{-1}$.

We then split $\mathbf{B}'$ into two parts as well, $\mathbf{B}_U$ (up) and $\mathbf{B}_D$ (down), both 3×6 ,

$$\mathbf{B}' = \begin{bmatrix} \mathbf{B}_U \\ \mathbf{B}_D \end{bmatrix}, \quad (1.145)$$

and we will demonstrate below that $\mathbf{B}_U$ is irregular and $\mathbf{B}_D$ regular. The matrix $\mathbf{N}$ of equation (1.142) is evaluated at the surface and keeps the same structure as $\mathbf{Y}$ (left part regular, right irregular), so that performing the products in equation (1.142) results in

$$\mathbf{N}\mathbf{B}'\mathbf{I}_c = [\mathbf{N}_R \ \mathbf{N}_I] \begin{bmatrix} \mathbf{B}_U \\ \mathbf{B}_D \end{bmatrix} \mathbf{I}_c = \mathbf{N}_R\mathbf{B}_U\mathbf{I}_c + \mathbf{N}_I\mathbf{B}_D\mathbf{I}_c. \quad (1.146)$$

In this way we have separated $\mathbf{N}\mathbf{B}'\mathbf{I}_c$ into the two terms $\mathbf{N}_R\mathbf{B}_U\mathbf{I}_c$ and $\mathbf{N}_I\mathbf{B}_D\mathbf{I}_c$, which present a different behavior since

$$\lim_{l \rightarrow \infty} \mathbf{N}_R\mathbf{B}_U\mathbf{I}_c \rightarrow +\infty, \quad \text{whereas} \quad \lim_{l \rightarrow \infty} \mathbf{N}_I\mathbf{B}_D\mathbf{I}_c \rightarrow 0. \quad (1.147)$$

The same holds for $\mathbf{M}\mathbf{B}'\mathbf{I}_c$ in equation (1.142) and we are thus allowed to write

$$\lim_{l \rightarrow \infty} \mathbf{N}\mathbf{B}\mathbf{I}_c \rightarrow \mathbf{N}_R\mathbf{B}_U\mathbf{I}_c \quad \text{and} \quad \lim_{l \rightarrow \infty} \mathbf{M}\mathbf{B}\mathbf{I}_c \rightarrow \mathbf{M}_R\mathbf{B}_U\mathbf{I}_c. \quad (1.148)$$

As a result, when we determine the products in equation (1.142) by taking into account equation (1.148), we obtain

$$\lim_{l \rightarrow \infty} \mathbf{K}_{lj} \rightarrow \left(\frac{\mathbf{N}_R\mathbf{B}_U\mathbf{I}_c \cdot (\mathbf{M}_R\mathbf{B}_U\mathbf{I}_c)^\dagger}{\frac{d}{ds} \det(\mathbf{M}_R\mathbf{B}_U\mathbf{I}_c)} \right)_{s=s_j} \cdot \mathbf{b}. \quad (1.149)$$

The relaxation times result from the approximated form of the secular equation (1.140):

$$\det[\mathbf{M}_R\mathbf{B}_U\mathbf{I}_c] = 0, \quad (1.150)$$

with $\mathbf{B}_U$ defined as

$$\mathbf{B}_U = \mathbf{Y}^{(1)U-1}(r_2) \prod_{i=2}^{N-1} \mathbf{Y}^{(i)}_R(r_i) \mathbf{Y}^{(i)U-1}(r_{i+1}). \quad (1.151)$$

We can demonstrate that $\mathbf{B}'$ keeps the same structure as in equation (1.145) (upper part irregular, lower part regular) independently of the number of layers. It is possible to rewrite equation (1.141) by grouping the matrices evaluated at the same radii and by isolating the one at the CMB:

$$\mathbf{B}' = \left[\prod_{i=2}^{N-1} \mathbf{Y}^{(i-1)-1}(r_i) \mathbf{Y}^{(i)}(r_i) \right] \mathbf{Y}^{(N-1)-1}(r_c). \quad (1.152)$$

It is evident, then, that all couples of matrices evaluated at the same radii compensate each r_i^l term with an analogous r_i^{-l} term: the only net dependence on the radius is from the inverse fundamental matrix at the CMB. As a consequence, the general structure of $\mathbf{B}'$ is the same as $\mathbf{Y}^{-1}(r_c)$, thus characterized by an irregular upper part and a regular lower one (as is clear from the structure of the fundamental matrix and from $\mathbf{Y}\mathbf{Y}^{-1} = \mathbf{I}$).

A further refinement of this last result is also possible. A direct consequence of equation (1.152) is the possibility of applying only a partial approximation to equation (1.141), instead of the full one in equation (1.151), by splitting the product into two parts:

$$\prod_{i=2}^{NA-1} \mathbf{Y}^{(i-1)-1}(r_i) \mathbf{Y}^{(i)}(r_i) \prod_{i=NA}^{N-1} \mathbf{Y}_U^{(i-1)-1}(r_i) \mathbf{Y}_R^{(i)}(r_i), \quad (1.153)$$

where NA is the first boundary where approximated solutions are used. In this way a good compromise can be found between approximation accuracy and numerical stability. As a general rule, the resolution required by shallow seismic events can be reached by approximating boundaries only within the lower mantle, typically represented by the CMB itself.

The possibility of a partial approximation becomes crucial to implementing the case of an internal mantle loading as a pointlike seismic source in the crust or lithosphere: these applications will be shown in Chapter 8, dealing with post-seismic deformation.

All the arguments presented continue to hold and no further demonstration is required when the approximation is applied below the source: the new terms only consist of matrices evaluated at the surface and at the source depth and are thus not affected by the approximation.

9.2 RESCALING THE SOLUTION

As anticipated above, the net dependence of the amplitudes on the basis functions in the $\mathbf{B}'$ matrix results from the inverse fundamental matrix evaluated at the CMB.

Considering that only the first column of $\mathbf{I}_c$ depends on powers of the radius (see equation (1.103)), due to the free-slip boundary condition at the interface with the fluid core, it is easy to show that

$$\mathbf{N}_R \mathbf{B}_U \mathbf{I}_c \cdot (\mathbf{M}_R \mathbf{B}_U \mathbf{I}_c)^\dagger (:) \begin{bmatrix} r_c^{-2l} & r_c^{-2l} & r_c^{-2l} \\ r_c^{-2l} & r_c^{-2l} & r_c^{-2l} \\ r_c^{-2l} & r_c^{-2l} & r_c^{-2l} \end{bmatrix} \quad (1.154)$$

and

$$\det(\mathbf{M}_R \mathbf{B}_U \mathbf{I}_c) (:) r_c^{-2l} \quad (1.155)$$

so that, once we make use of equations (1.154) and (1.155) to evaluate equation (1.149), the dependence on r_c^{-2l} compensates between the numerator and the denominator.

The key point consists in the possibility of taking the $r^{\pm l}$ dependence out of the computation at the very beginning and to obtain the same residues. This can be done by normalizing $\mathbf{Y}_R$ by means of r^l and $\mathbf{Y}_U^{-1}$ by means of r^{-l} . In this way the stiffness problem is avoided.

The accuracy of the approximation is analyzed for the value of elastic and fluid Love numbers for an incompressible Earth in the case of surface loading. All the results are expressed as normalized residuals

$$k_{norm.res.} = \left| \frac{k - k_{app.}}{k} \right|, \quad (1.156)$$

with k the Love number without the approximation and $k_{app.}$ the approximated Love number. In the figures, a square represents the radial displacement number, a triangle the tangential displacement and a circle the gravitational ones.

The earth model taken into consideration is characterized by five layers: an elastic lithosphere 120 km thick, a viscoelastic mantle with chemical discontinuities at 420 km and 660 km and, finally, an inviscid core. Values for rigidity and density are PREM-averaged (Dziewonski and Anderson, 1981) as in Table 2.2 of the following chapter and viscosity in the mantle is fixed at 10^{21} Pa s.

In Figure 1.7 we show the elastic k normalized residues of the Earth when the approximation procedure is started at different depths. In panel (a) we take the regular part of the solution only at the core-mantle boundary: differences between exact and approximated Love numbers are up to 40% but affect only the first few terms and are negligible for degree 10 and higher. In panel (b) we rescale all the boundaries from the 420 km discontinuity downwards (i.e. also at 660 km and the CMB): the vertical and gravitational responses are highly affected (more than 100% difference) but still converge quite rapidly to the exact solution (1% difference at degree 50); the horizontal Love number is

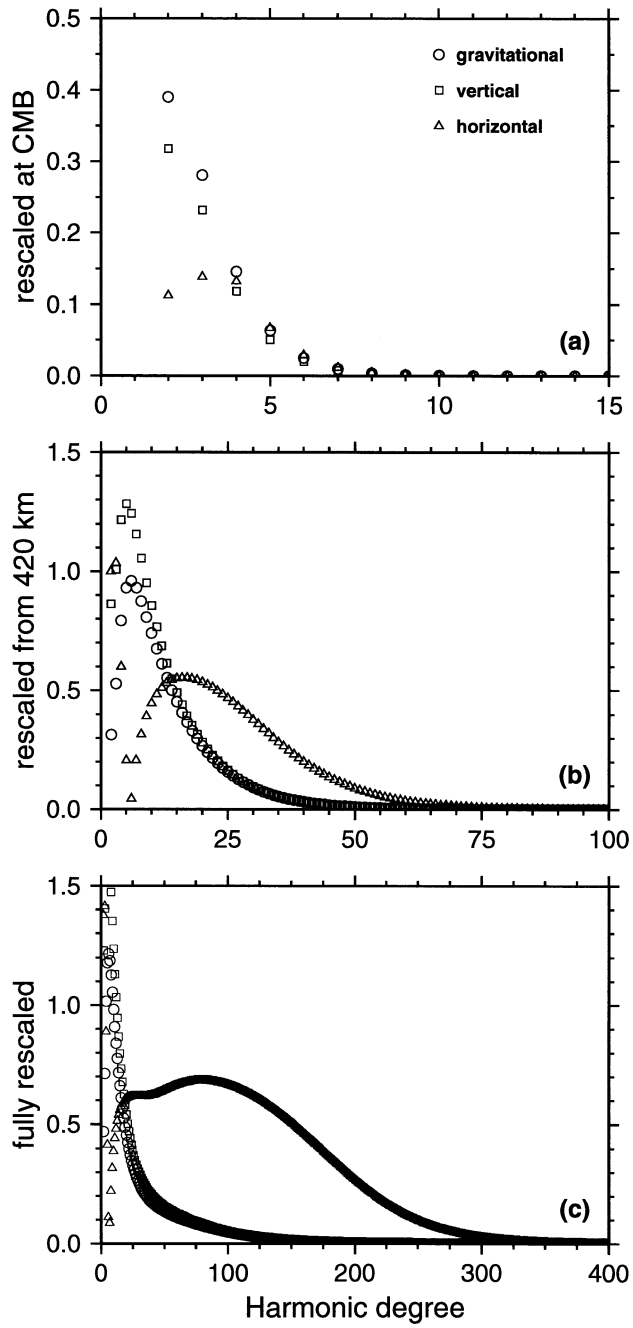


Figure 1.7. Normalized residual elastic Love numbers (Figure 1 in Riva and Vermeersen, 2002).

less affected, but convergence is slower (less than 1% at degree 70). Finally, in panel (c), full approximation is applied: more terms are required before reaching convergence between the exact and the approximated solution (1% difference around degree 160 for the vertical and the gravitational numbers), in particular for the horizontal Love number which remains significantly different for the first 200 terms and reaches the 1% level around degree 330.

In Figure 1.8 we show the effect of the approximation for the fluid response: results are not significantly different from the elastic case. When only the CMB is approximated, as in panel (a), convergence is very fast and the maximum difference is limited (only the gravitational number is above 10% for degree 2). When approximation is started at 420 km, as in panel (b), both vertical and gravitational Love numbers show a discrepancy from the exact solution smaller than 20% and again a fast convergence (below 1% at degree 25), whereas a few more horizontal terms are highly affected. This last feature, however, is due to the fact that the horizontal fluid number is changing sign around degree 10 and the normalization is not appropriate. The only limit we found is the impossibility of applying the approximation at a boundary with an elastic layer when the fluid response is evaluated: null residuals are obtained in this case. This fact, however, does not represent a real problem, since several thousand terms are already reached by just rescaling the solution at deeper boundaries. Another important issue is the numerical computation of spherical harmonics using the recursive relation for the Legendre polynomials: results up to degree 60,000 have been compared after working with both double- and quadruple-precision and the relation appears to be stable. The stiffness problem can thus be overcome by applying a successful approximation and rescaling procedure to the analytical propagator matrix technique commonly employed in normal mode models. The irregular fundamental solutions in non-homogeneous earth models can be neglected, the weight getting smaller and smaller with increasing harmonic degree. Keeping only the regular part of the fundamental solution makes it possible to by-pass the non-commutativity problem of the matrix algebra and to rescale the solution within the propagation procedure. Thus, the method presented here is an alternative to the minors-only method commonly employed in (seismic) numerical normal mode modeling.

These results provide a quantitative estimate of the applicability of the approximation procedure. In practice, in most cases it is only necessary to rescale the CMB, which means that just the first few terms need the full solution in order to obtain adequate results. However, even when the solution is fully approximated, good results are obtained after a few hundred terms.

This approach has been explicitly discussed for the case of an incompressible linear rheology due to the availability of a short explicit form for both the fundamental matrix and its inverse. This assumption is not necessary for the

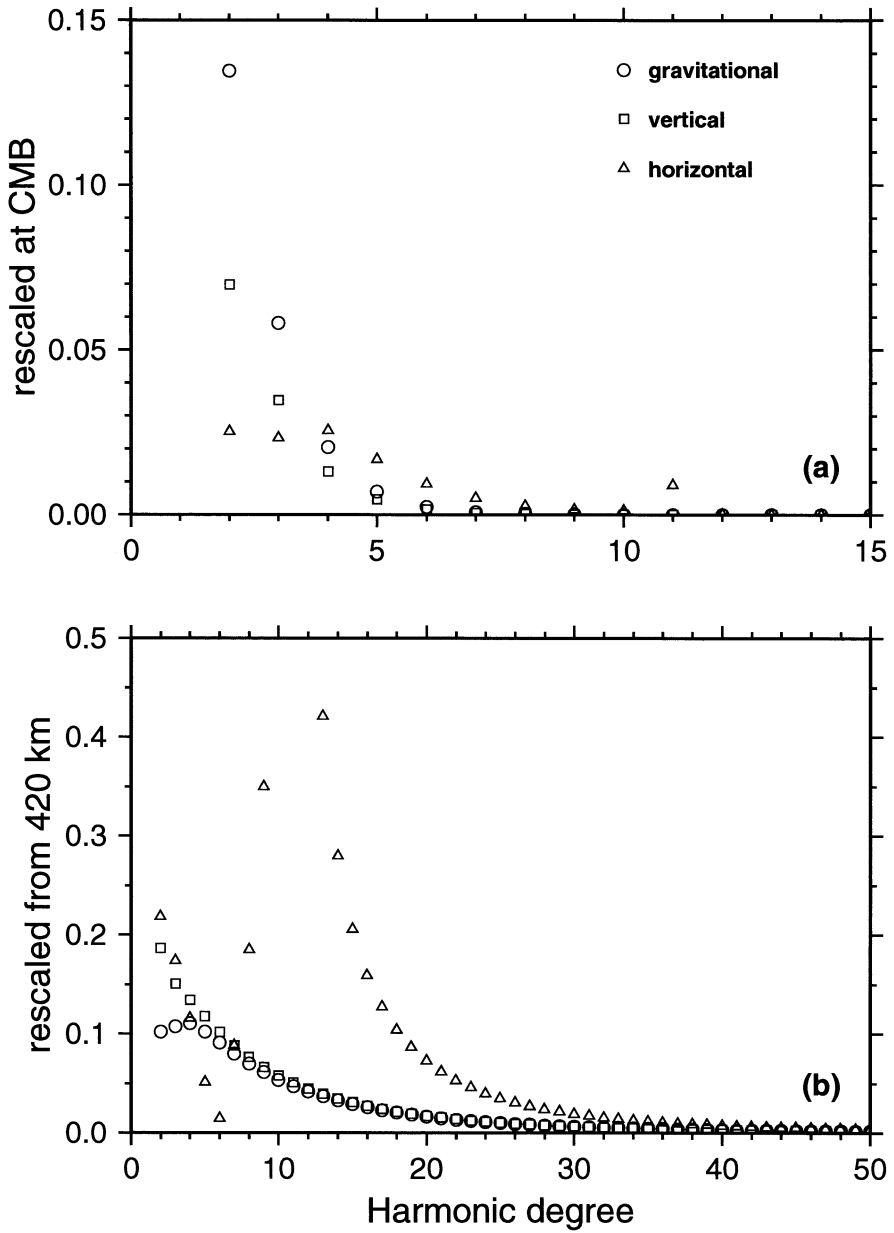


Figure 1.8. Normalized residual fluid Love numbers (Figure 2 in Riva and Vermeersen, 2002).

demonstration, as the character of the fundamental solutions (regular/irregular) is the only information which is required.

Chapter 2

MULTI-LAYER MODELS

1. ANALYTICAL DEVELOPMENT OF STRATIFIED VISCOELASTIC SPHERICAL EARTH MODELS

Multi-layer, spherically stratified, self-gravitating relaxation models with a large number of layers (more than 100) can be dealt with analytically once the limits posed by the procedures based on standard routines to detect a small number of modes used in several analyses (Spada *et al.*, 1992b; Piersanti *et al.*, 1995) are avoided by means of the bisection algorithm for detecting the modes in the s -domain. In order to let the reader become accustomed with the analytical methodologies in the normal mode approach, relaxation processes are studied in this chapter based on Vermeersen and Sabadini (1997) for both Heaviside surface loads and tidal forcings. Simulations of the relaxation process of a realistic earth model with an incompressible Maxwell rheology show that models containing about 30 to 40 layers have reached continuum limits on all timescales and for all harmonic degrees up to at least 150 whenever an elastic lithosphere is present, irrespective of the viscosity profile in the mantle. Especially fine-graded stratification of the shallow layers proves to be important for high harmonic degrees in these models. The models produce correct long-time (fluid) limits. It is shown that differences in transient behavior of the various models are due to the applied volume-averaging procedure of the rheological parameters.

Some doubts have arisen about the correctness or completeness of discrete normal mode models as employed in viscoelastic relaxation modeling (Hanyk *et al.*, 1995; Fang and Hager, 1995). Hanyk *et al.* (1995) argue for the existence of ‘non-modal’ contributions, becoming most apparent in the fluid limits they obtain for the h Love number, representing the Green function for the radial displacement given by equation (1.137) for V_l , in a comparison between a normal

mode model and a direct time integration model, while Fang and Hager (1995) argue that the discrete normal mode model is not valid when the rheological parameters are continuous functions of radius. In this chapter we show that these purported shortcomings in the fundamental physics of (discrete) normal mode theory are artificial consequences of numerical inaccuracies, theoretical mis-interpretations and of using incomplete sets of normal modes. We show explicitly that the models based on the normal mode technique in Chapter 1 produce both continuous behavior, resulting from continuous rheological stratifications, and discrete behavior, resulting from sharp density contrasts, as at the outer surface and the core mantle boundary. The differences between volume-averaged models and fixed-boundary contrast models are outlined. Reducing many-layer models with a volume-averaging procedure before employing a normal mode analysis is both economical and highly accurate on all timescales and for all spherical harmonic degrees. The procedure minimizes chances of missing contributing modes, while using models with more layers will not result in any substantial increase in accuracy.

In Chapter 1, it was explicitly shown that development and building of analytical models containing a large number of layers are practically possible; here we go up to models containing more than 100 layers. Formulas for these models have been first expressed in Vermeersen *et al.* (1996a) in a concise form, where it was shown that simulations have high accuracies and many aspects that remain elusive in numerical models become understandable. A nice example to illustrate this may be found in the purely compressible relaxation (or dilatation) modes that are excited in self-gravitating spherical Maxwell models: numerical integration of the differential equations in the time-domain does not reveal anything about these modes. Wu and Peltier (1982) left the whole region in which they occur in the Laplace domain blank in their numerical normal mode models, as this region contains a myriad of modes that are hardly distinguishable from one another by conventional rootfinding procedures. Han and Wahr (1995) were able to include these myriad modes in their numerical normal mode model, showing that the region in which they occur is of importance because of the presence of viscoelastic modes which have significant strength. From their results, Han and Wahr (1995) deduced the conjecture that the dilatation modes might form denumerably infinite sets for the viscoelastic layers. Vermeersen *et al.* (1996b) were able to prove this conjecture by deriving a highly-accurate series approximation formula (their expression (46)) from the fundamental solutions resulting from their analytical normal mode model. It can also be deduced from that formula that the compressible modes are annihilated in the incompressible limit and thus are truly dilatation modes.

Although models employing straightforward numerical integration in the time domain are more versatile than the analytical normal mode models (e.g., it is relatively easy to incorporate lateral variations and non-linear rheologies in

such models compared to normal mode models), they can hardly match normal mode models in deducing these kinds of features of the physics of relaxation. In this respect, analytical normal mode models should certainly be considered with respect to numerical integration models as complementary: normal mode models are able to produce reliable results for relatively crude models, they are ideal as tests for numerical codes incorporating lateral variations and non-linear rheologies and, most importantly, they provide an understanding and justification of the basic physics of the relaxation process.

In Vermeersen *et al.* (1996a), it was shown that the concept of the existence of Hanyk *et al.*'s (1995) non-modal contribution, apart from modal contributions in normal mode modeling, cannot be substantiated by an analytical normal mode analysis. The fluid limits for the Love numbers for radial and lateral displacement and for the gravity potential showed the correct values for 30-layer models and their volume-averaged 5-layer counterparts. Differences were found between the 5-layer and 30-layer models on transient timescales, however, and these were interpreted by Vermeersen *et al.* (1996a) as being due to the applied volume-averaging procedure. A proof of this conjecture could not be given, however, although a reason was given by referring to the fact that there is a harmonic degree-dependent sensitivity in the Love numbers for the depth-dependence of the layering, which is not taken into account in the volume-averaging procedure. A point that accompanies this is the question of when this depth-dependence resulting from the volume-averaging procedure is not visible any more in the Love numbers, or: how many layers are needed to achieve continuum limits? This point is of more than mere theoretical importance. In the past, some have used only 5 layers at most (e.g., Spada *et al.*, 1992b), others 160 (J.X. Mitrovica, personal communication) or even 200 (Han and Wahr, 1995) in their normal mode models. If it could be found that discrepancies on transient timescales are indeed solely due to the volume-averaging procedure as proposed by Vermeersen *et al.* (1996a), thus refuting Fang and Hager's (1995) invalidity claim, and if including more layers would not result in any increase of accuracy for the relaxation process for a certain number of layers, then some kind of standard layering could be used for future models and eventually for benchmarks. Reducing very fine-graded models or continuous profiles to a standard layering by means of a volume-averaging procedure before the normal mode method is applied would not only be more economical with respect to computer time, but would also decrease the chances of making errors like missing contributing relaxation modes. The main purpose of this chapter thus consists of two main themes: determining the modeling accuracy for a range of models and understanding the causes for the discrepancies on transient timescales. By 'transient timescales' we mean timescales between the elastic and fluid timescales, that is, times at which the curves are deviating from the elastic limit and have not reached the fluid limit. In the figures on the Love

numbers as a function of time this can usually be seen very easily: most curves will move from one horizontal line on short timescales (the elastic limit) to another horizontal line on long timescales (the fluid limit). The non-horizontal behavior of the curves in between the two horizontal lines depicts the transient behavior.

The advantage of an analytical secular determinant, as given in equations (1.106) or (1.112), is that there are no numerical inaccuracies present before the rootfinding procedure is applied to it. It depends solely on the accuracy of the method by which the roots are extracted from the secular determinant whether all the roots are found, whether ‘false’ roots show up, and what the accuracy of the values is.

Once the roots (poles in the complex plane, as equation (1.133) indicates), or inverse relaxation times of the various relaxation modes, have been found, the strengths of the modes can be determined analytically by complex contour integration, as shown by equation (1.134). If the complete set of modes and accompanying strengths have been found, the solutions for the model can be easily determined via equation (1.137).

Although the analytical method as presented in Chapter 1 might look fairly complete, there remain some gaps:

(1) as far as we know, there exists at the moment no analytical proof that all poles are of first order, as implicitly assumed by equation (1.136). We simply assume here that they are all of first order, as we do not find any indication that some of them would be of a higher order. If poles of a higher order were present in our models, they would lead to infinities in the residues. A simple example might illustrate this. Suppose that in the secular determinant given by equation (1.106) the term $(z - z_0)^{-2}$ can be factorized so that there would be a second order pole at $z = z_0$. If we were mistakenly to assume that it is a first order pole, then we would have the residue of the pole as proportional to

$$\lim_{z \rightarrow z_0} (z - z_0) \times \frac{1}{(z - z_0)^2}, \quad (2.1)$$

which would be infinitely large. In all the runs that we have done, we have never encountered such a situation (an exception concerns the root $s = 0$ and the roots equal to the negative inverse Maxwell times, but these are of no importance - see further on). On the contrary, all fluid Love numbers determined as a combination of the strengths of the modes resulted in sensible values, as will be seen by the hundreds of curves in the following figures. It is of some importance to note that when lateral variations are taken into account, higher-order poles do show up (D’Agostino *et al.*, 1997).

(2) The secular determinant is an analytical function but cannot be solved analytically when the earth model has more than two layers.

(3) An analytical proof that each boundary between two viscoelastic layers excites one buoyancy mode and two viscoelastic modes has been given by Wu and Ni (1996). The two viscoelastic modes disappear whenever the Maxwell times on both sides of the boundary are equal. However, the systematic pattern in which they seem to occur (Han and Wahr, 1995, page 297) is not yet understood, although circumstantial evidence is strong.

Vermeersen *et al.* (1996b) have derived highly accurate approximate series solutions for the fundamental solutions for the dilatation modes (purely compressible relaxation modes) of simple compressible Maxwell models.

The results shown in this section are based on one of the simplest ways in which the roots of an analytical function can be determined: grid-spacing in combination with a bisection algorithm. When a change of sign of the analytical function is detected by the grid-spacing procedure, the bisection algorithm determines the position of the root with a very high accuracy (we always use quadruple precision as a standard). It can happen that, when two roots are very close to each other in the Laplace domain, the spacing of the grid is too large to detect them (the analytical function changes from negative to positive, and again to negative in a region smaller than the grid). Such a situation can be detected as we know beforehand how many roots we can expect. For example, if we are dealing with a 30-layer incompressible model in which the mantle is situated between a homogeneous lithosphere and a homogeneous inviscid core, then there are 28 viscoelastic mantle layers. If all the mantle layers have different rheological parameters, then we can expect a total of 84 roots: each boundary layer between two viscoelastic layers produces 3 roots (a buoyancy mode and 2 viscoelastic modes - see Han and Wahr (1995) and Vermeersen *et al.* (1996a)), which amounts to $3 \times 27 = 81$ roots; the outer surface, CMB and lithosphere-uppermost mantle layer boundary each produces a buoyancy mode.

After having run a lot of simulations, we have come up with the following.

- When an elastic lithosphere is present and the model does not contain a lot of layers (about 30), for low harmonic degrees we usually find all the modes by the standard values we employ for the grid-sizes. For high harmonic degrees, the grid-sizes for the domain where the viscoelastic modes occur often have to be taken smaller to detect all the modes, as for these higher degrees some pairs of modes are very closely packed. This is also the case for those models which contain a lot of layers (56 and 111 in our case).
- When the lithosphere is viscoelastic instead of elastic, the situation is further complicated by the notorious instabilities that arise for very small values of the Laplace domain parameter s (Wu and Peltier, 1982; for a fluid the system of 6 fundamental solutions is overdetermined), but this problem is ameliorated by the following:

- the long-term buoyancy modes do not carry any noticeable strength at all (see also Han and Wahr, 1995). Typical values for the strength of these small long-term s -value modes are less than 10^{-10} . Even for a 111-layer model, the total strength of the long-term modes is thus smaller than 10^{-7} , being negligible compared with the strengths of the strongest modes, which are in the range $10^{-1} - 10^{-2}$.

Finding the roots is thus most difficult for those models which do not contain an elastic lithosphere, have a large number of layers and for high harmonic degrees. But, on the other hand, the accuracies of models which do not contain an elastic lithosphere are the easiest ones to determine because of the fluid limits they produce for the k Love number. The closer these fluid limits are to -1 , the higher the accuracy is. For our 5-layer models with a viscoelastic lithosphere having a viscosity of 10^{21} Pa s (thus resulting in 11 relaxation modes), we typically reach accuracies for the fluid k Love number which deviate less than 10^{-12} per cent from -1 for the low degree harmonics, which is close to machine precision. This accuracy drops with increasing number of layers and harmonic degree deviations of 1-3 per cent from -1 in the case of some of our models which contain 111 layers for harmonic degree 150. In the case of this latter accuracy, the fluid limit is clearly not reached. Another reason for the difficulty of attaining a high accuracy for high harmonic degrees is that the viscoelastic modes have longer relaxation times when the harmonic degree is larger. Thus, in the case where an elastic lithosphere is absent, the viscoelastic modes shift closer to $s = 0$ and, consequently, into a highly unstable domain. Although the results here are derived from incompressible models, there is no fundamental difference with respect to questions like whether or not ‘non-modal contributions’ are present when the Lamé parameter λ is taken as finite (compressible models). Vermeersen *et al.* (1996b) have shown that compressibility mainly has two effects: shifting the relaxation times of the modes of the corresponding incompressible model (and thus altering the modal strengths), and introducing denumerably infinite sets of dilatation modes which can be approximated by analytical formulas. These denumerably infinite sets of modes intermingle with the buoyancy and viscoelastic modes, and this can lead to trouble with (numerical) rootfinding procedures. However, dilatation modes have negligible strength, as shown by Vermeersen *et al.* (1996b). It is thus a matter of finding all the contributing buoyancy and viscoelastic modes in these compressible models amidst the dilatation modes. In this respect these dilatation modes are no more than a nuisance to rootfinding procedures. From a physical point of view they are quite interesting, however: e.g., these modes imply that compressibility comes not as a continuous spectrum but as a discrete spectrum for finite-volume models. The analytical formulas given by Vermeersen *et al.* (1996b) can be quite helpful in, for example, separating them from the other modes, or filtering the dilatation modes out, retaining the same (but shifted

in time and altered in strength) modes as in the incompressible cases we deal with here. This will make it easier for rootfinding procedures as applied to compressible models to find all the contributing modes.

Fang and Hager (1995) point to the fact that at the inverse Maxwell time the compliant Lamé parameters of a Maxwell model in the Laplace domain become singular, and dub the consequences of this fact for normal mode theory the 'singularity mystery'. In an analytical approach these singularities give no problems, on the contrary, they illuminate some aspects of the relaxation process: in Spada *et al.* (1992b) one can find that for a 5-layer model the three inverse Maxwell times (the shallow upper mantle, transition zone and lower mantle) are part of the roots of an 18th-degree polynomial (see equations (6), (7) and (8) in Spada *et al.*, 1992b). These three modes associated with the inverse Maxwell times of the three viscoelastic mantle layers (the lithosphere is elastic and the core inviscid) do not carry any strength and are most likely exactly equal to zero, as can be proven for one-layer and two-layer models. As was already pointed out in the discussion on the gaps in the analytical method, here we may have a situation in which there potentially are poles of higher order: if a layer contains the same numerical values for rigidity and viscosity as another layer, then the Maxwell times for the two layers are thus the same, resulting in a second-order root of the secular determinant coinciding with the negative inverse Maxwell time (according to the term in square brackets of equation (6) in Spada *et al.*, 1992b). But one should also keep in mind that such 'Maxwell modes' are (almost certainly) non-existent, i.e. they most likely have a strength which is exactly zero. The same applies to the null roots $s = 0$, which would have a sixth-order pole in a 5-layer model (Spada *et al.*, 1992b). Although not being analytical proofs, the former notions thus agree with the strong circumstantial evidence one gets by plotting the secular function that the singularities at the various inverse Maxwell times are essentially removable discontinuities, i.e. the limits from the left and right towards a discontinuity seem to be the same, although a stringent analytical proof of this conjecture is still lacking.

The term 'continuous poles' which Fang and Hager (1995) coined does not imply that their model is not discretized. In fact, they are using a Runge-Kutta scheme for solving their continuous model, and acknowledge on page 850 (left column) of their paper that a "Runge-Kutta propagation is a method of using discretized layers to approximate a continuous function. One may reasonably argue that we are dealing with discrete modes by using Runge-Kutta propagation while we are thinking of 'continuous modes'."

Fang and Hager (1995, page 850 lower right column) discuss the possibility of the existence of second- or higher-order mode contributions on transient timescales, even when these contributions are lacking in the fluid limit, but they do not explicitly state that the fluid limits in their models are equivalent to

the values obtained in hydrostatic equilibrium. It is thus unclear to us whether they refrain from Hanyk *et al.*'s (1995) results with respect to purported non-modal contributions in the fluid limit (see also Vermeersen *et al.*, 1996a). In this respect it is regrettable that neither Fang and Hager (1995) nor Hanyk *et al.* (1995) present modeling results on the k Love number for models which do not have an elastic lithosphere. Whatever method one is using, for low harmonic degrees one should always get -1 to very high precision in the fluid limit for all surface loading models. Although PREM contains density inversions in the shallow part of the earth model (Dziewonski and Anderson, 1981), it is very easy to delete these inversions temporarily for testing purposes (moreover, we will show that the layers which have these density inversions do not contribute significantly to the Love numbers for low harmonic degrees). Although not a guarantee for correctness, it is a guarantee for incorrectness if one does not find the -1 fluid limit for low harmonic degrees.

1.1 UNIFORM MANTLE VISCOSITY AND ELASTIC LITHOSPHERE

To gain some insight into the effects of stratification on relaxation, we shall start with the following simple model. An elastic lithosphere with a thickness of 120 km surrounds the viscoelastic mantle having a viscosity of 10^{21} Pa s. The elastic parameterization and density profile of the five cases are all based on PREM (Dziewonski and Anderson, 1981). These five cases have not been selected at random; all have special meanings.

Five-layer models have been used in the past in the context of analytical viscoelastic relaxation modeling (e.g., Spada *et al.*, 1992b; Ricard *et al.*, 1993; Vermeersen *et al.*, 1994). There is one important difference, however, in comparison with the stratification that is used between the 5-layer models in this section and the one that has been used in the past. The elastic parameterization and density profile were based on what we will dub from now on fixed-boundary contrast models (e.g., Sabadini *et al.*, 1982a; Yuen *et al.*, 1982). The densities for the layers (lithosphere, shallow upper mantle, transition zone, lower mantle) in these models were chosen in such a way that the final assemblage contained density contrasts which agreed with the density contrasts given by PREM for the interfaces. Thus, for example, the density contrast at the boundary between the transition zone and the lower mantle is about 10 per cent in the fixed-boundary contrast model, agreeing with the value PREM gives, but the lithosphere has, according to Table 2.1, a density of 4120 kg/m^3 , which is far greater than the volume-averaged density of the lithosphere as given in Table 2.2 (in fact, the highest density that PREM gives for a layer in the top 120 km is less than 3381 kg/m^3). It is important to realize that the 5-layer models in the following are, unless stated explicitly otherwise, volume-averaged models. The density con-

Table 2.1. Parameters for the 5-layer fixed-boundary contrast Earth model. r is the distance with respect to the center of the Earth, ρ the density of the layer, and μ the rigidity.

layer	r (km)	ρ (kg/m ³)	μ (N/m ²)	
1	6371 – 6271	4120	7.28×10^{10}	lithosphere
2	6271 – 5951	4120	9.54×10^{10}	shallow upper mantle
3	5951 – 5701	4220	1.10×10^{11}	transition zone
4	5701 – 3480	4508	1.99×10^{11}	lower mantle
5	3480 – 0	10925	0	inviscid fluid core

Table 2.2. Parameters for the 5-layer volume-averaged earth model. r is the distance with respect to the center of the Earth, ρ the density of the layer, and μ the rigidity.

layer	r (km)	ρ (kg/m ³)	μ (N/m ²)	
1	6371 – 6250	3184	6.0243×10^{10}	lithosphere
2	6250 – 5951	3434	7.2666×10^{10}	shallow upper mantle
3	5951 – 5701	3857	1.0639×10^{11}	transition zone
4	5701 – 3480	4878	2.1944×10^{11}	lower mantle
5	3480 – 0	10925	0	inviscid fluid core

trasts at the boundaries are thus not fixed to the values PREM gives as they were in all 5-layer models, which, as we know, have been published in the past.

For our models, we have taken the PREM values in Tables II and III of Dziewonski and Anderson (1981; p. 312-315). In these tables, PREM contains 55 layers between the Core Mantle Boundary (CMB) and the surface. Of these 55 layers there are 12 (of which one only partly) inside the lithosphere, which leaves 43 layers for the mantle. If the core is taken into account as an inviscid layer, then we have 56 layers for a homogeneous core-mantle and lithosphere PREM, while this is reduced to 45 layers when the lithosphere is also taken as homogeneous. Thus we have included the 45-layer and 56-layer cases in our models. In the 30-layer model the lithosphere is taken as homogeneous, while every 3 mantle layers of PREM in succession (except for the 4 layers nearest to the core: these layers are volume-averaged as pairs) are reduced by the volume-averaging procedure to two layers. In the 111-layer model the 55 layers of PREM for the mantle and lithosphere are doubled. Thus we end up with a minimal number of five cases (5, 30, 45, 56 and 111 layers), which we can posit as indicative of changes that can be expected for any other number of layers.

It is not clear beforehand that the fineness of the stratification will result in the same kind of accuracies for all types of forcings. In fact, it might well be that in the case of an ice load on the Earth's surface it is more important to stratify

the shallow parts of the Earth more finely than in the case of the tidal interaction between the Earth and the Moon. Therefore we will study the relation between surface loading and stratification as well as tidal forcing and stratification.

Figure 2.1 contains the temporal evolution of the Love numbers for the case of a unit mass load which is positioned at time $t = 0$ at the surface. Note that we make use of the same symbols for surface loads and tidal loading in the Love numbers, specifying each time the kind of forcing that we are considering.

For such a Heaviside-type of forcing the Love numbers read, according to equation (1.137) for the $\mathbf{K}_l$ vector,

$$(ag/M_E)(h_l/g, l_l/g, (1 + k_l)) = \mathbf{K}_{le} - \sum_{j=1}^M \frac{\mathbf{K}_{lj}}{s_j} \left(1 - e^{s_j t} \right), \quad (2.2)$$

where k_l stands for the Love number of the gravitational potential, h_l for the radial displacement and l_l for the horizontal displacement. The terms $\mathbf{K}_{lj}$ are the individual modes j with negative inverse relaxation times s_j . The quotient $\mathbf{K}_{lj}/s_j$ is called the strength of mode j . $\mathbf{K}_{le}$ denotes the elastic limit.

The fluid limit of the Love numbers is easily obtained from

$$\mathbf{K}_{lf} = \lim_{t \rightarrow \infty} \mathbf{K}_l(t) = \mathbf{K}_{le} - \sum_{j=1}^M \frac{\mathbf{K}_{lj}}{s_j}. \quad (2.3)$$

Figure 2.1a shows plots of equation (2.2) for the gravity potential, Figure 2.1b for the radial displacement, and Figure 2.1c for the horizontal displacement. Each of the Figures 2.1a-c consists of eight plots depicting the cases for spherical harmonic degrees 2, 6, 15, 30, 50, 75, 100 and 150. Each plot shows the curves for the cases of 5, 30, 45, 56 and 111 layers. The five numbers inside each plot caption the layering.

The first aspect that immediately strikes the eye upon glancing over the various plots is that the curves virtually cover each other or fall into two categories. Generally, for low degrees all five curves huddle together, while for high degrees the curves for the cases of 5, 30 and 45 layers become separated from the cases of 56 and 111 layers. This subdivision into two groups agrees with a subdivision into models which have a uniform lithosphere and an internal lithospheric layering. Apparently, the higher the harmonic degree is, the more sensitive the relaxation becomes for layering inside the lithosphere. The fact that even for high harmonic degrees the curves for the cases of 56 and 111 layers almost completely cover each other indicates, however, that it does not matter very much whether such an internally stratified lithosphere consists of 24 layers or the volume-averaged equivalent of 12 layers. Saturated continuum limits up to harmonic degree 150 are apparently reached for 56-layer models with 12 internal elastic lithospheric layers for a uniform viscosity profile. The

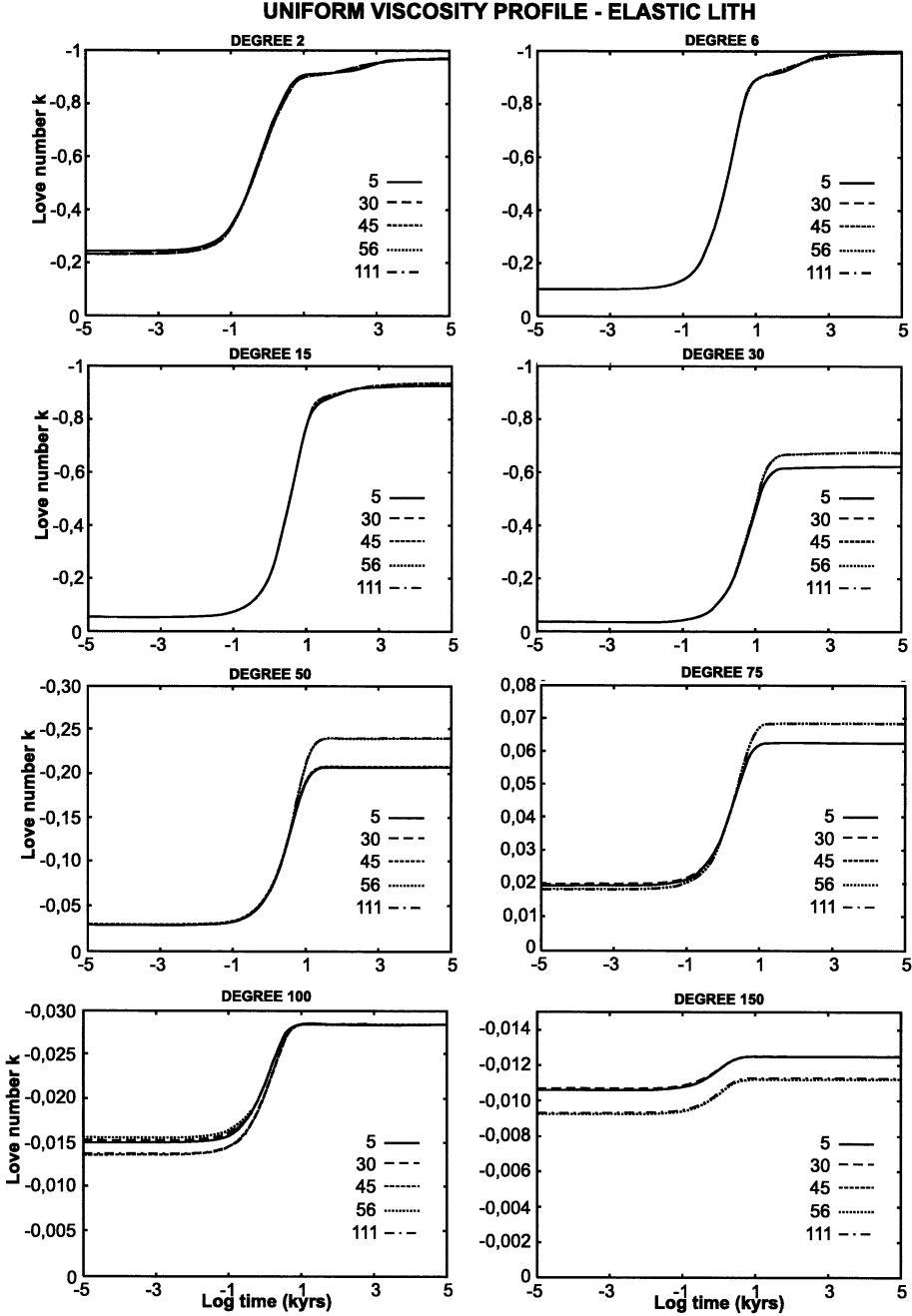


Figure 2.1a. Temporal evolution of the Love numbers k , h and l following the positioning of a Heaviside load at the surface at time $t = 0$ for the case of a 10^{21} Pa s mantle and an elastic lithosphere: (a) shows the gravitational potential k Love number for the spherical harmonic degrees 2, 6, 15, 30, 50, 75, 100 and 150, as indicated in each panel on the top. Each of the eight panels contains five curves delineating the temporal behavior for the 5-, 30-, 45-, 56- and 111-layers stratifications. The densities and rigidities for the models are derived from PREM.

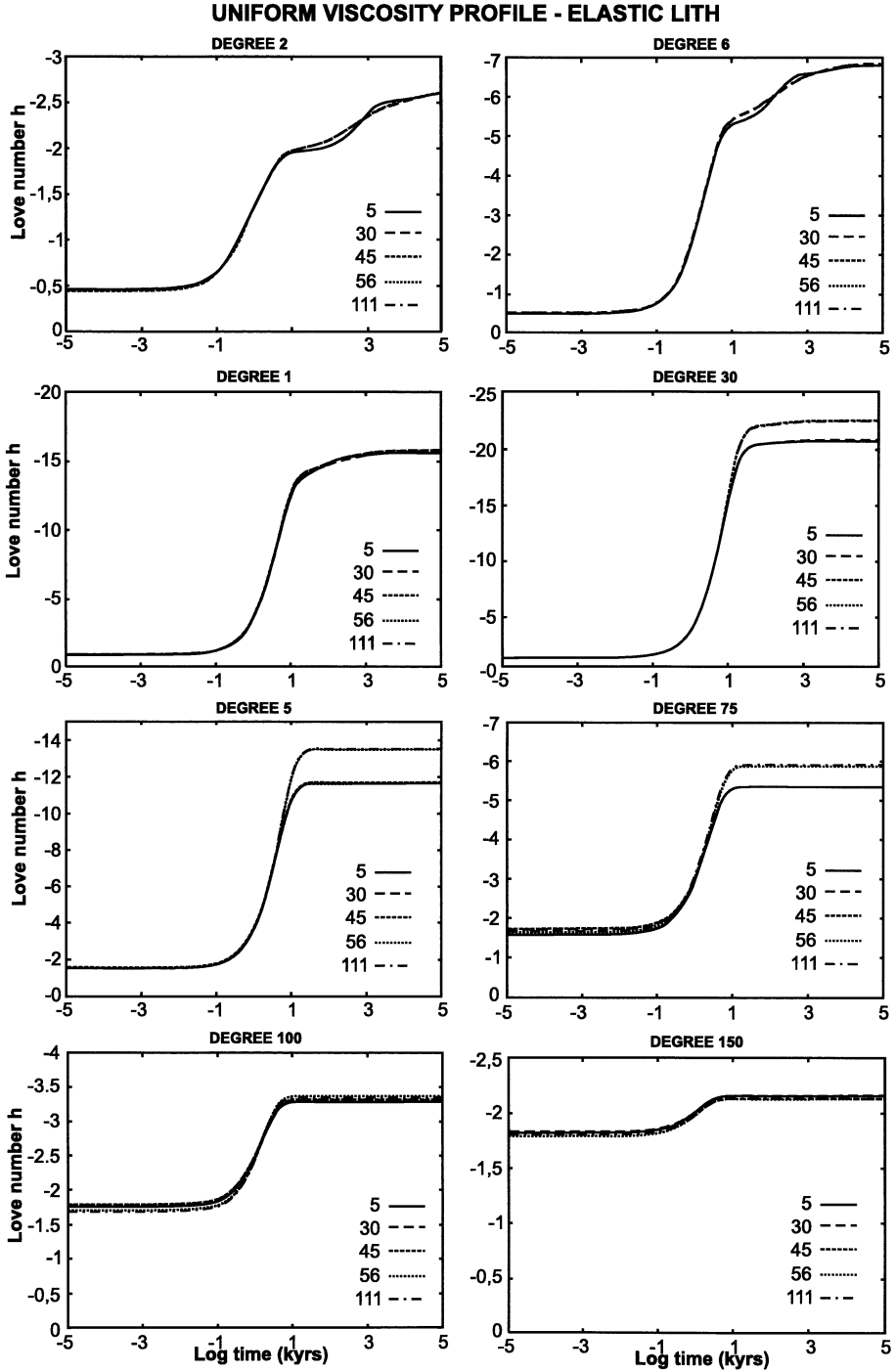


Figure 2.1b. The same as (a), but now for the vertical deformation Love number h .

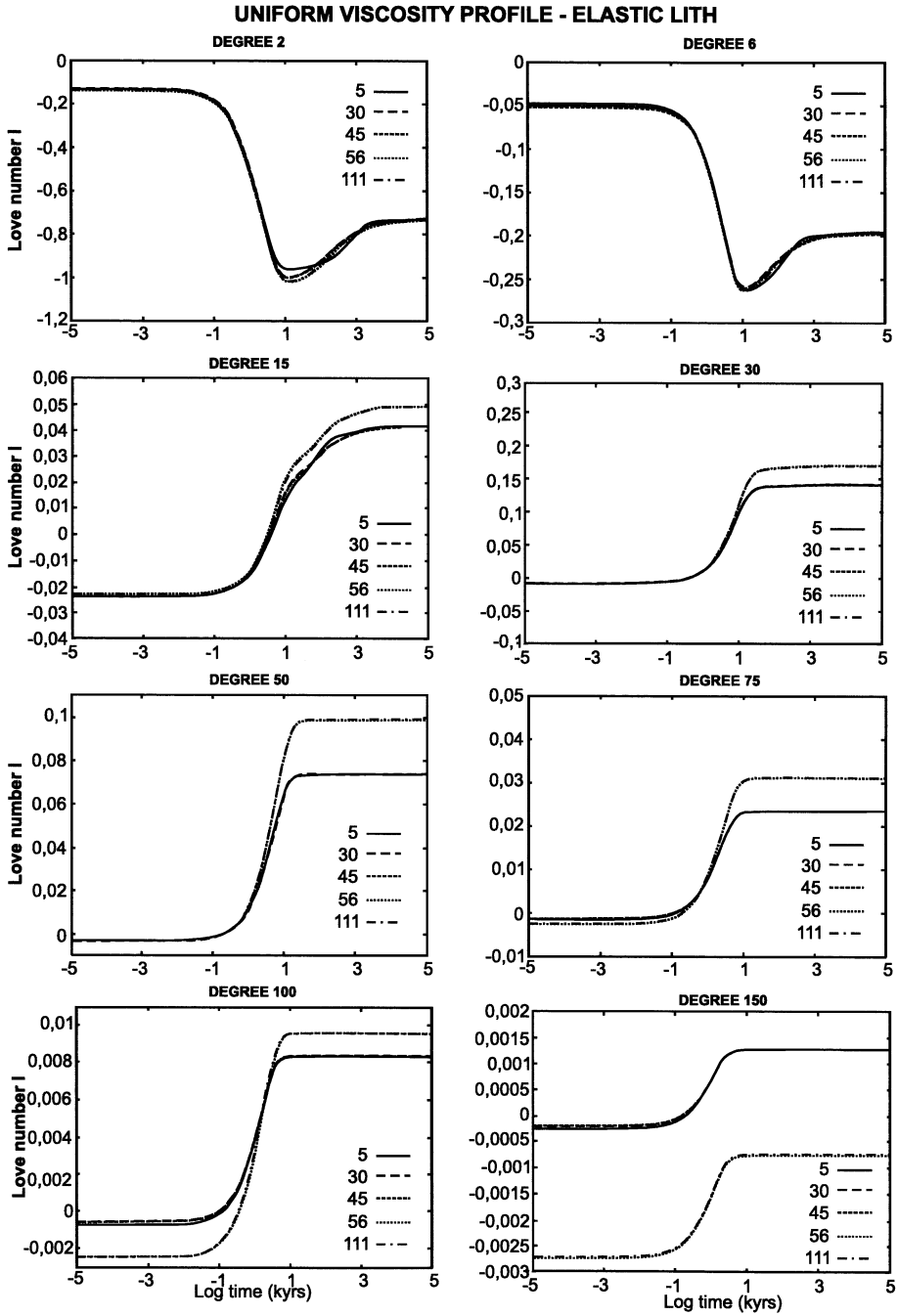


Figure 2.1c. The same as (a) and (b), but for the horizontal deformation Love number l .

fact that, upon closer examination, the 30-layer models do not seem to give any significantly different response on all timescales from the 45-layer models indicates that models with 30 mantle layers produce saturated continuum limits. Already the 5-layer model responses come close to the 30-layer model responses, but on transient timescales there are clear differences visible for the low degree harmonics 2, 6 and 15.

Figure 2.2 contains the same models as in Figure 2.1 but now for the case of a tidal forcing instead of a surface load.

The same general conclusions as were reached for the surface load models can be drawn for these tidal forcing models. Closer examination of the plots in Figure 2.2 do reveal some differences with the tendencies in Figure 2.1, however. Whereas in Figure 2.1a in the panels for harmonic degrees 2 and 6 the k Love numbers of the 5-layer model do not differ in the fluid limit from the other four cases, they clearly do in Figure 2.2a for the case of a tidal forcing. This is also the case for degree 2 in the case of the h Love number (compare Figures 2.1b and 2.2b). There are some differences visible also for the harmonic degrees 100 and 150 in the case of the h Love number, but this might be largely due to the fact that the h Love numbers in the high harmonic plots in Figure 2.2b are quite small (compare the vertical scales with the corresponding plots in Figure 2.1b). The behavior of the l Love numbers in Figure 2.2c does not differ much from the behavior of the plots in Figure 2.1c as far as the deviations among the various curves is concerned. Only for degree 2 can these differences be seen to be somewhat larger in the case of a surface load than in the corresponding case of a tidal forcing.

The conclusion that can be drawn from this first collection of models is that a model containing about 30 to 40 layers shows saturated continuum behavior for all timescales and for all harmonic degrees up to degree 150 when the mantle has a uniform viscosity and the lithosphere is elastic, no matter whether one wishes to study the relaxation due to surface loading or to tidal forcings. If one has an initial model containing more layers (e.g., for density stratification), it is economical not only first to reduce this stratification to a 30- to 40-layer model by a volume-averaging procedure and then run the normal mode model, but doing so also reduces the great risk of not detecting all the contributing modes in the rootfinding procedure.

Having scrutinized the homogeneous mantle viscosity case, it now becomes interesting to see what a convex mantle profile will give.

1.2 CONVEX MANTLE VISCOSITY AND ELASTIC LITHOSPHERE

The convex viscosity profile in Figure 2.3 is based on Figure 2 in Ricard and Wuming (1991).

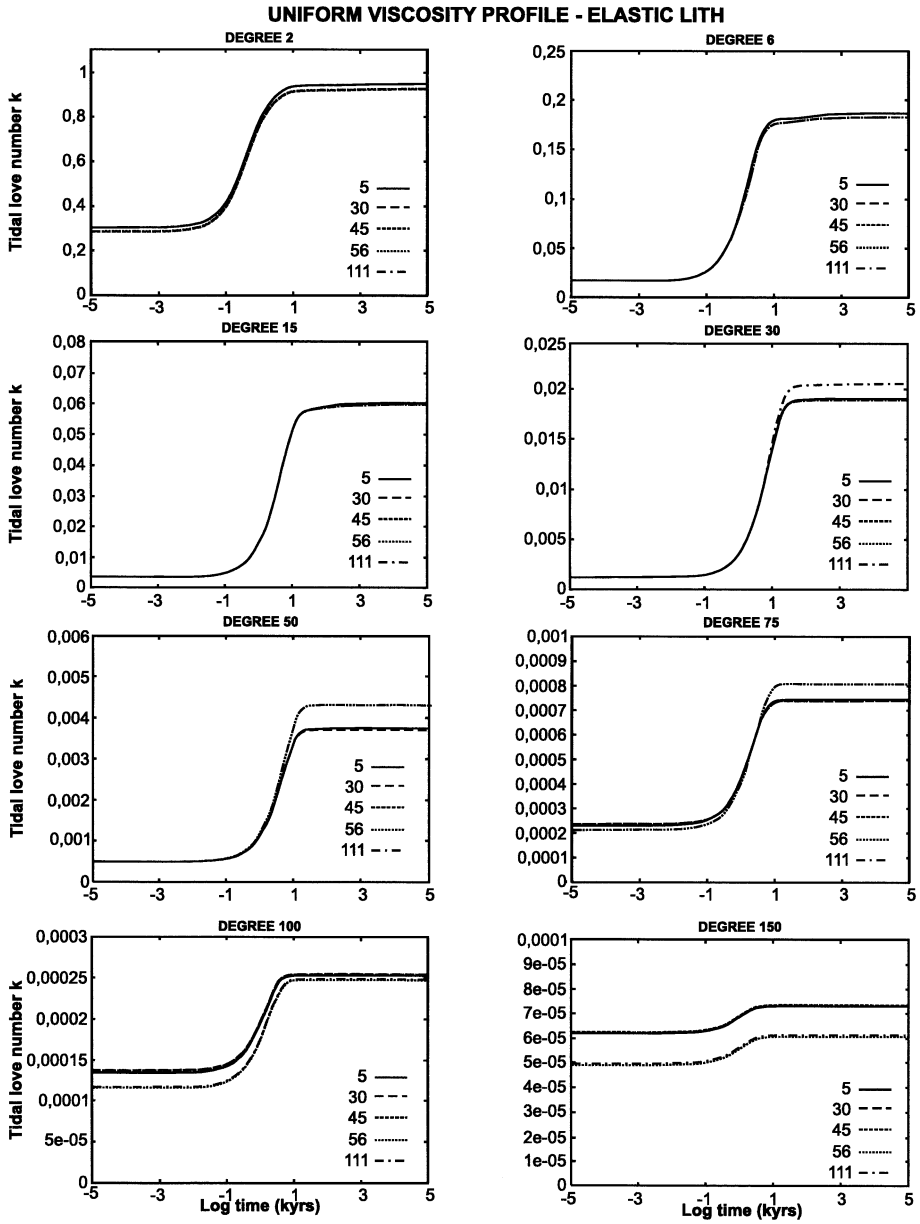


Figure 2.2a. As in Figure 2.1, but for tidal forcing instead of the Heaviside surface load. k Love number.

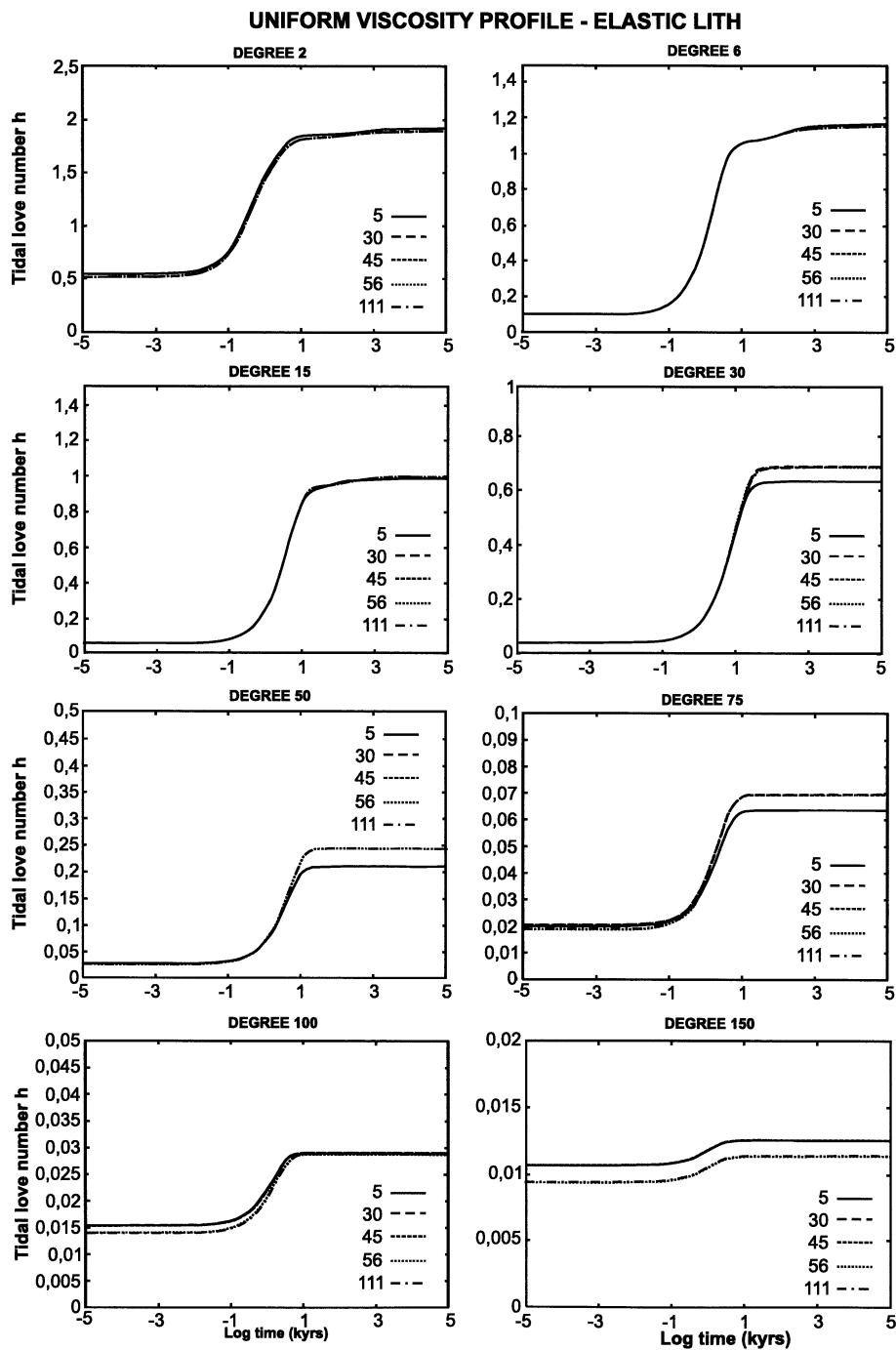


Figure 2.2b. As in panel (a) for the h Love number.

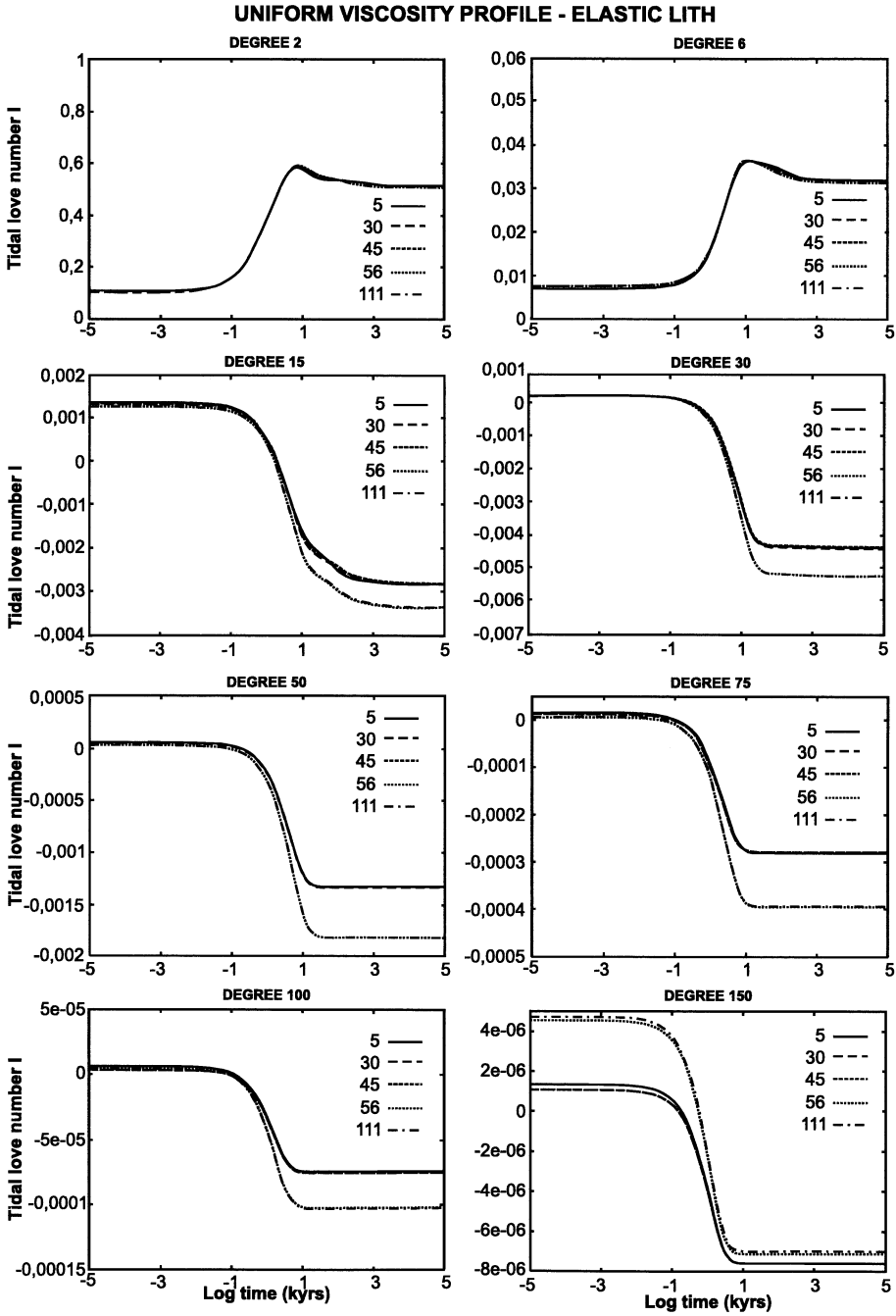


Figure 2.2c. As in panels (a) and (b) for the l Love number.

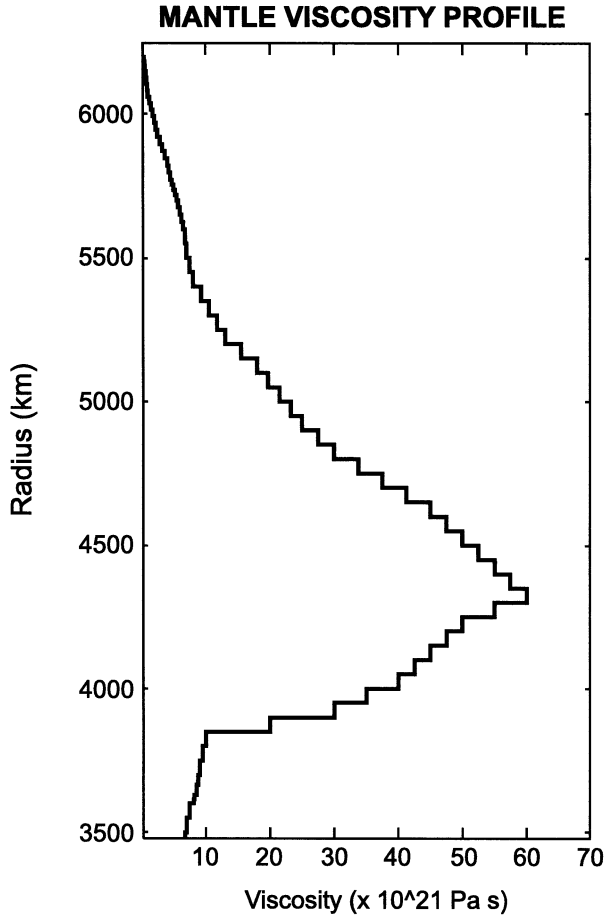


Figure 2.3. Viscosity profile of the mantle as a function of the radial distance to the center of the Earth for the convex case. The viscosity profile for the 111-layer case is drawn; for the other stratifications (5, 30, 45 and 56 layers) this profile has to be volume averaged according to the stratification (Figure 1 in Vermeersen and Sabadini, 1997).

Ricard and Wuming (1991) deduced their 15-layer radial mantle viscosity profile from a combined inversion of geoid, dynamic topography and plate velocities data. Their Figure 2 shows a gradual increase of viscosity from the top to the mid-lower mantle, followed by a gradual decrease to the CMB. There is no discontinuity in the viscosity profile at the upper mantle-lower mantle interface. The increase from the asthenosphere to the CMB is over more than two orders of magnitude. We have used their 15-layer profile to roughly interpolate a viscosity profile for our 111-layer model as depicted in our Figure 2.3. The 30-layer volume-averaged viscosity profile in Figure 2.3 was already used in the 30-layer model of Vermeersen *et al.* (1996a), where it was shown

that even a model with a convex mantle viscosity profile produces correct fluid limits. The profile in Figure 2.3 forms the basis for the determination of the viscosity profile of the 5, 30, 45 and 56 layer models by the volume-averaging procedure.

Figure 2.4 contains the plots for the same cases as in Figure 2.1, but now with the uniform mantle profile replaced by the convex viscosity profile in Figure 2.3.

The same general tendencies that were noticed in the former section are also visible in the panels of Figure 2.4, except for the behavior of the 5-layer cases. These 5-layer cases seem to differ somewhat more on transient timescales with the other layer cases in the panels of Figure 2.4 than in the corresponding plots of Figure 2.1. Also the fluid limits start to deviate more from the 30- and 45-layer cases than in Figure 2.1 for high harmonic degrees. Although in the fluid limit the 30- and 45-layer cases as a pair and the 56- and 111-layer cases as a pair match each other again very closely, on transient timescales they deviate slightly from each other for medium-large harmonic degrees. The internal differences in Figure 2.4 as compared with the (lack of) internal differences in Figure 2.1 must be due to the difference in viscosity profile, although these internal differences are not dramatic. Clearly, the volume-average procedure can lead to stronger differences in the case of a convex viscosity profile than in the case of a homogeneous viscous mantle, although again the small and often not distinguishable differences between the 30- and 56-layer curves as a pair and the 56- and 111-layer curves as a pair suggest that continuum limits are fairly well reached for models containing about 30 mantle layers and 10 lithosphere layers. Especially a comparison between the curves for harmonic degree 150 in Figures 2.1 and 2.4 shows these tendencies clearly (note the distinct difference of behavior between 0.1 kyrs and 10 kyrs of the 5-layer cases for the k Love number with respect to the 30- and 45-layer cases in both plots).

Our experience is that any varying viscosity profile with depth in the mantle of any kind will result in somewhat larger differences between, say, a 30-layer model and a 5-layer model than the corresponding differences in homogeneous viscosity profile models. The reason for this is most likely that such a varying viscosity profile leads to an extra averaging procedure. In a homogeneously viscous mantle, there is no need to average over the viscosity profile, as it will always result in the same viscosity value for all layers, no matter how small the layering is. For a varying viscosity profile with depth this is not so: the viscosity assigned to each layer by the volume-averaging procedure does depend on the number of layers of which the model consists. Where for a homogeneously viscous model only two parameters are affected by the volume-averaging procedure (density and elasticity), for any varying viscosity model there are three parameters which are affected (density, elasticity and viscosity). As the viscoelastic modes are dependent on the viscosity contrasts between

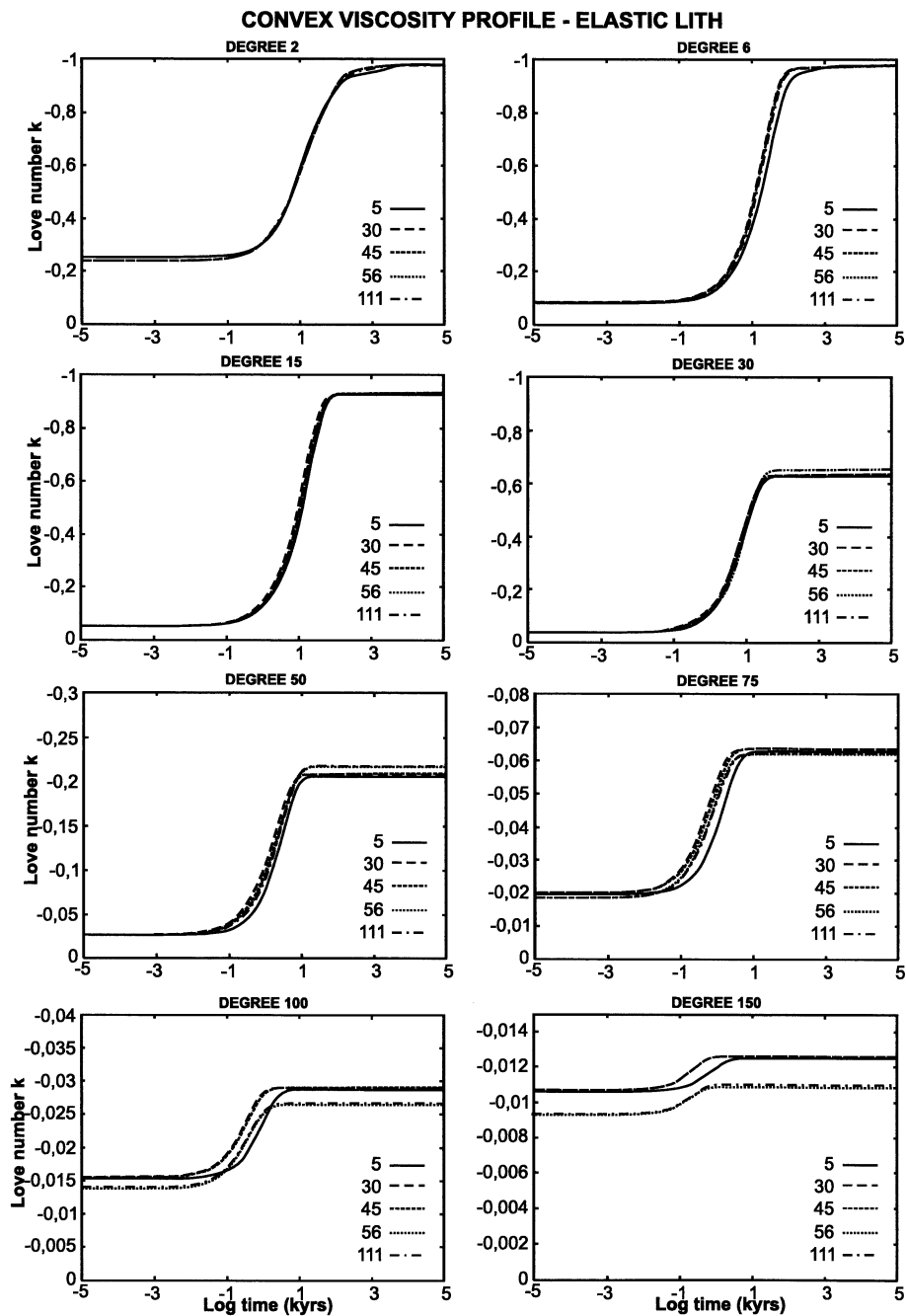


Figure 2.4a. As in Figure 2.1, except for the convex mantle viscosity profile of Figure 2.3. The lithosphere is again elastic.

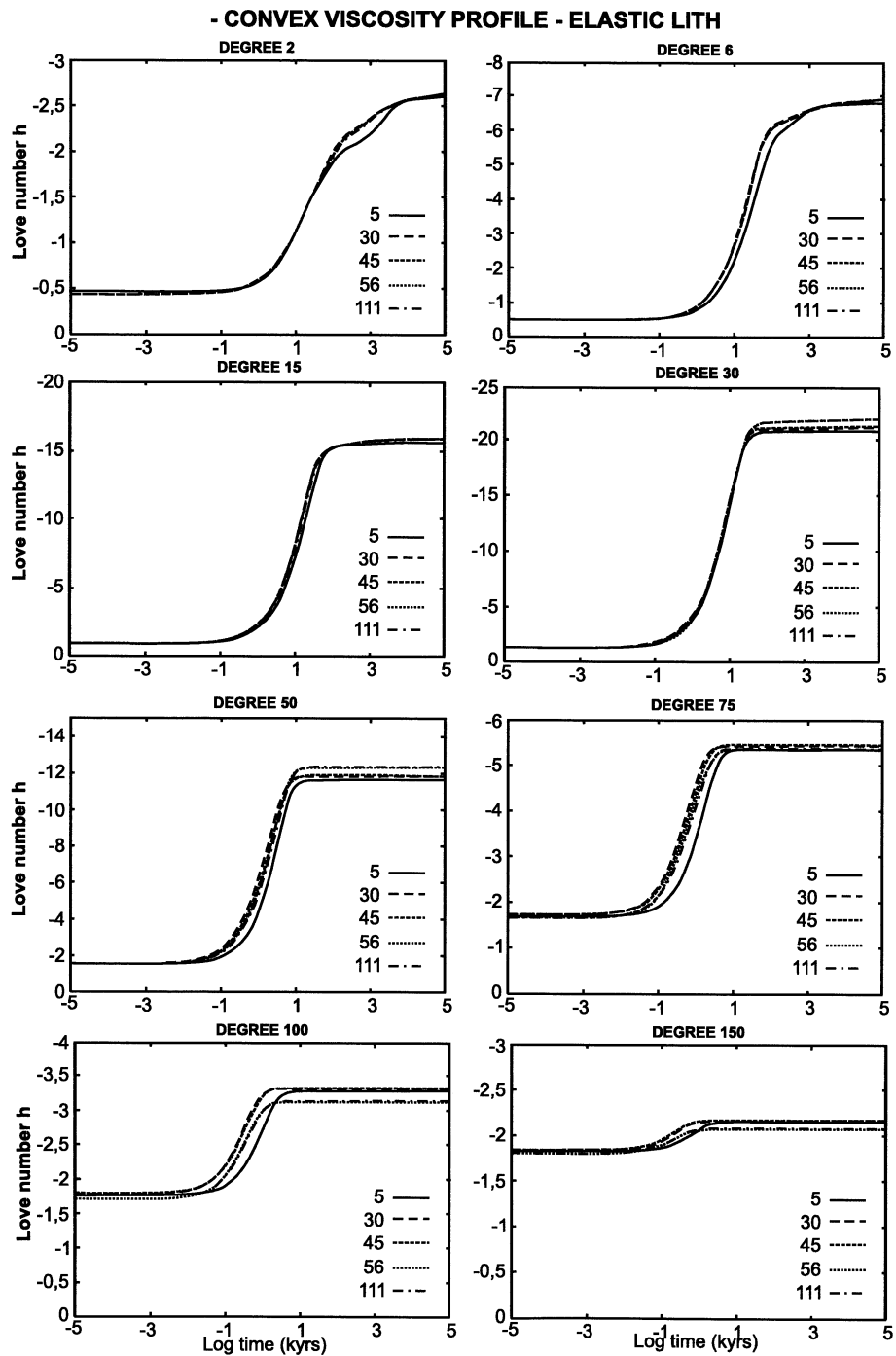


Figure 2.4b. (Continued)

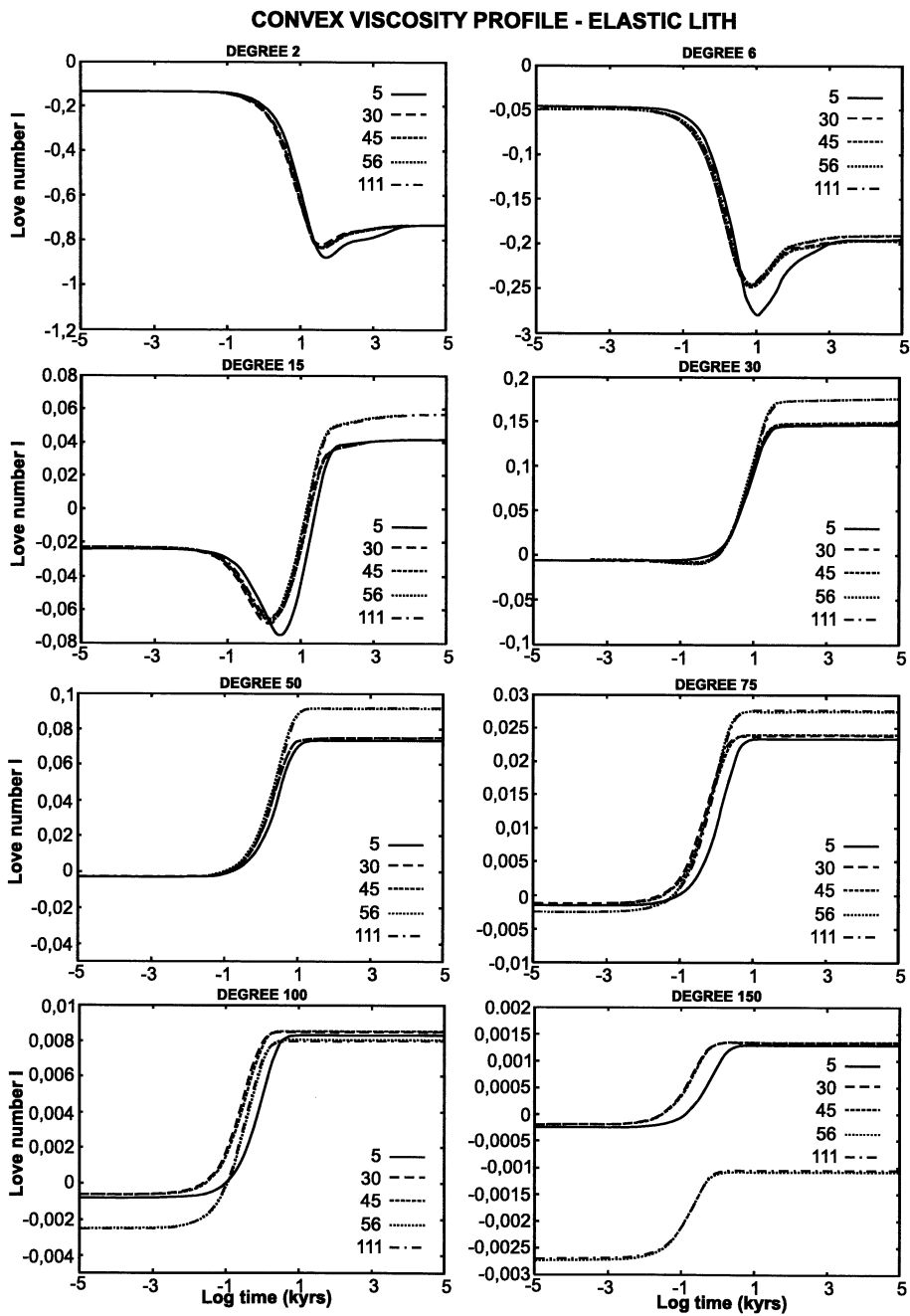


Figure 2.4c. (Continued)

the layers, we might thus expect that these viscoelastic modes will make the model results more dependent on the number of layers in the case of any varying viscosity profile than in the case of a homogeneous profile.

The fluid Love numbers are independent of the viscosity profile. So a comparison between the curves of the panels in Figure 2.4 and the corresponding curves in Figure 2.1 should ideally not result in differences. For low harmonic degrees this is indeed the case, as it is for all 5-layer models, but not for high harmonic degrees for the other layer models. There can be two reasons for this behavior: numerical inaccuracies or the buoyancy modes between 24.4 km and 220 km depth, which do carry considerable strength for high harmonic degrees. Note again in this respect that we have implemented PREM without annihilating the density inversions and that we have not taken the buoyancy modes resulting from these density inversions into account. The differences between the fluid limits in Figures 2.1 and 2.4 are most pronounced for the 56- and 111-layer cases, but also the 5-layer and 30-layer cases do not cover each other as well in Figure 2.4 as they do in Figure 2.1, although the 30-layer and 45-layer cases as a pair and the 56-layer and 111-layer cases as a pair do cover each other again very well, just as is the case in Figure 2.1. Increasing the accuracy of the rootfinder procedure did not result in any noticeable decrease in the differences in the fluid limits in all cases. Together with the fact that one would expect the influence of the region between 24.4 km depth and 220 km depth to become most pronounced at high harmonic degrees (in the 30-layer model the uppermost mantle layer has a lower density than the lithosphere; in the 45-layer model the two uppermost mantle layers have an inverse density profile), this would all point in the direction of the PREM density inversions as the likely cause for the discrepancies in the fluid limit. It thus becomes very interesting to consider what the fluid limits will be if the lithosphere is viscoelastic instead of elastic.

The same kinds of tendencies and differences occur in the cases of the tidal forcing depicted in Figure 2.5 when they are compared with the corresponding cases in Figure 2.2. Apparently, the fact that the 5-layer cases can differ in the fluid limit of the other layer cases for a tidal forcing, but not for a surface loading, is a general characteristic, independent of the mantle viscosity profile. Again, the various curves in Figure 2.5 illustrate the conclusion of Vermeersen *et al.* (1996a) that on intermediate timescales the 5-layer models and the corresponding 30-layer models can show significant differences. Figure 2.5c shows that even for low harmonic degrees these differences can be several tens of per cent. For high harmonic degrees the differences can amount to more than 100 per cent on all timescales between models with a stratified lithosphere and a volume-averaged homogeneous lithosphere. Again, it does not seem to matter much whether this stratification consists of 10 layers or 20 layers.

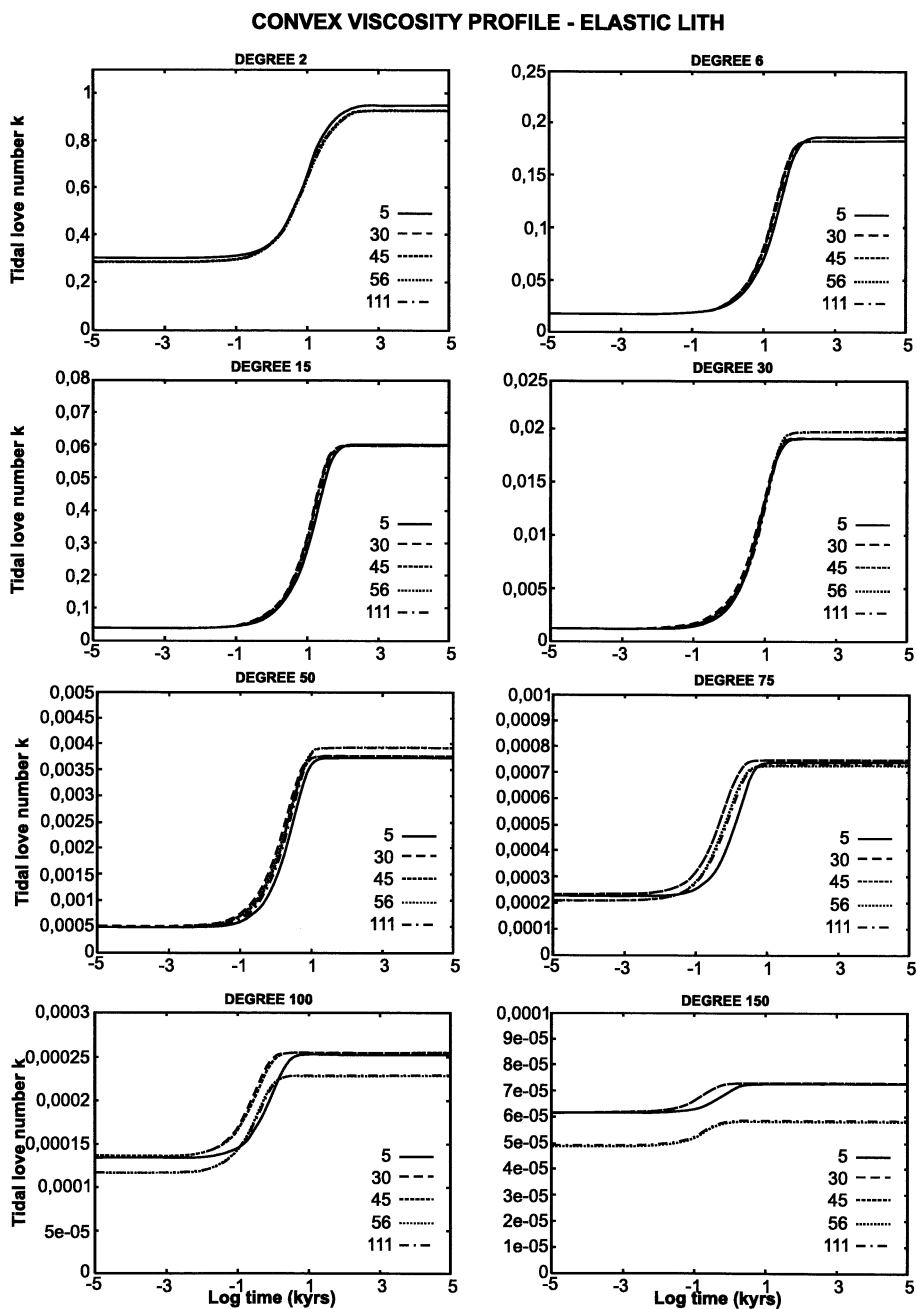


Figure 2.5a. As in Figure 2.4, except for the tidal forcing instead of the Heaviside surface load.

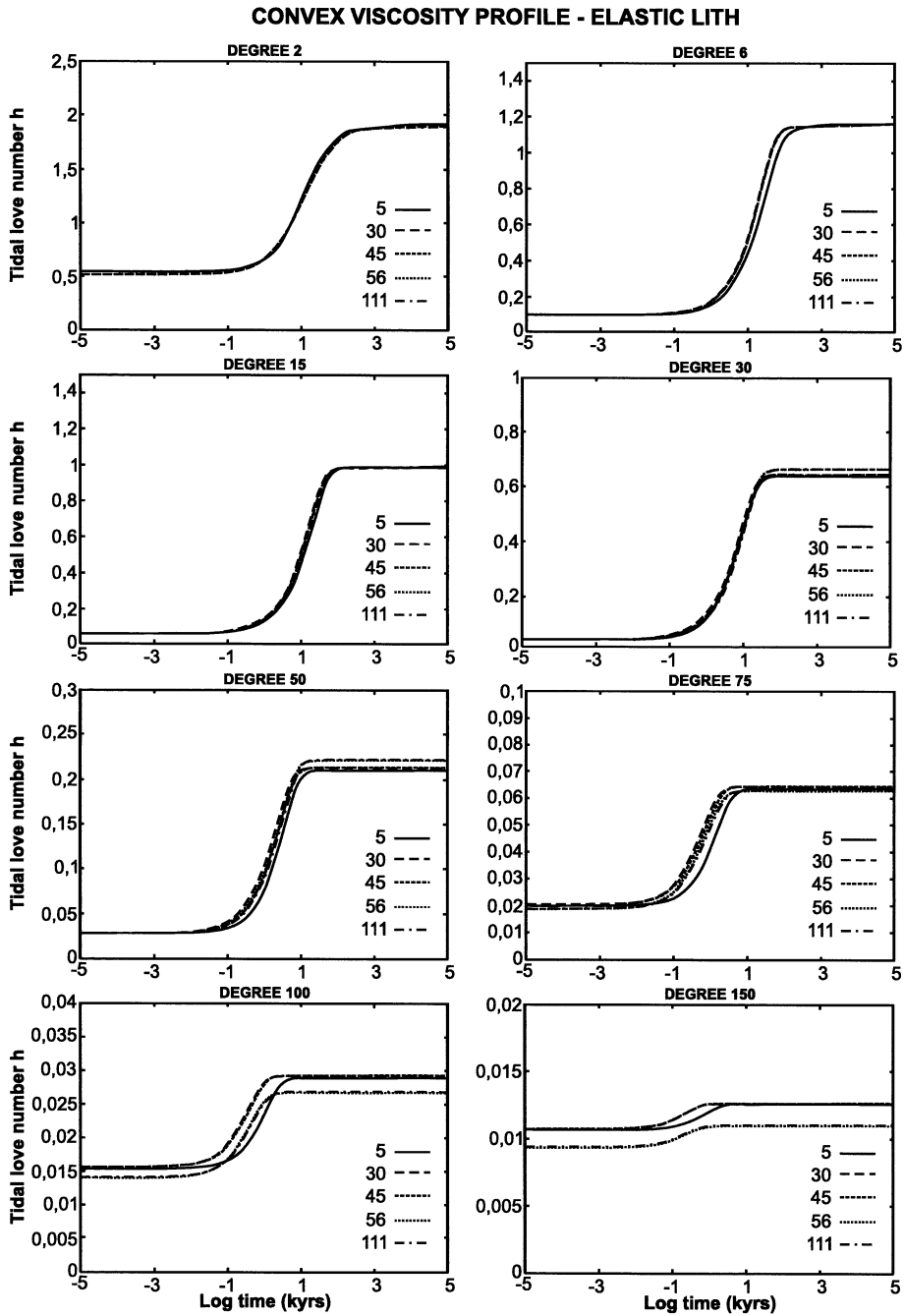


Figure 2.5b. (Continued)

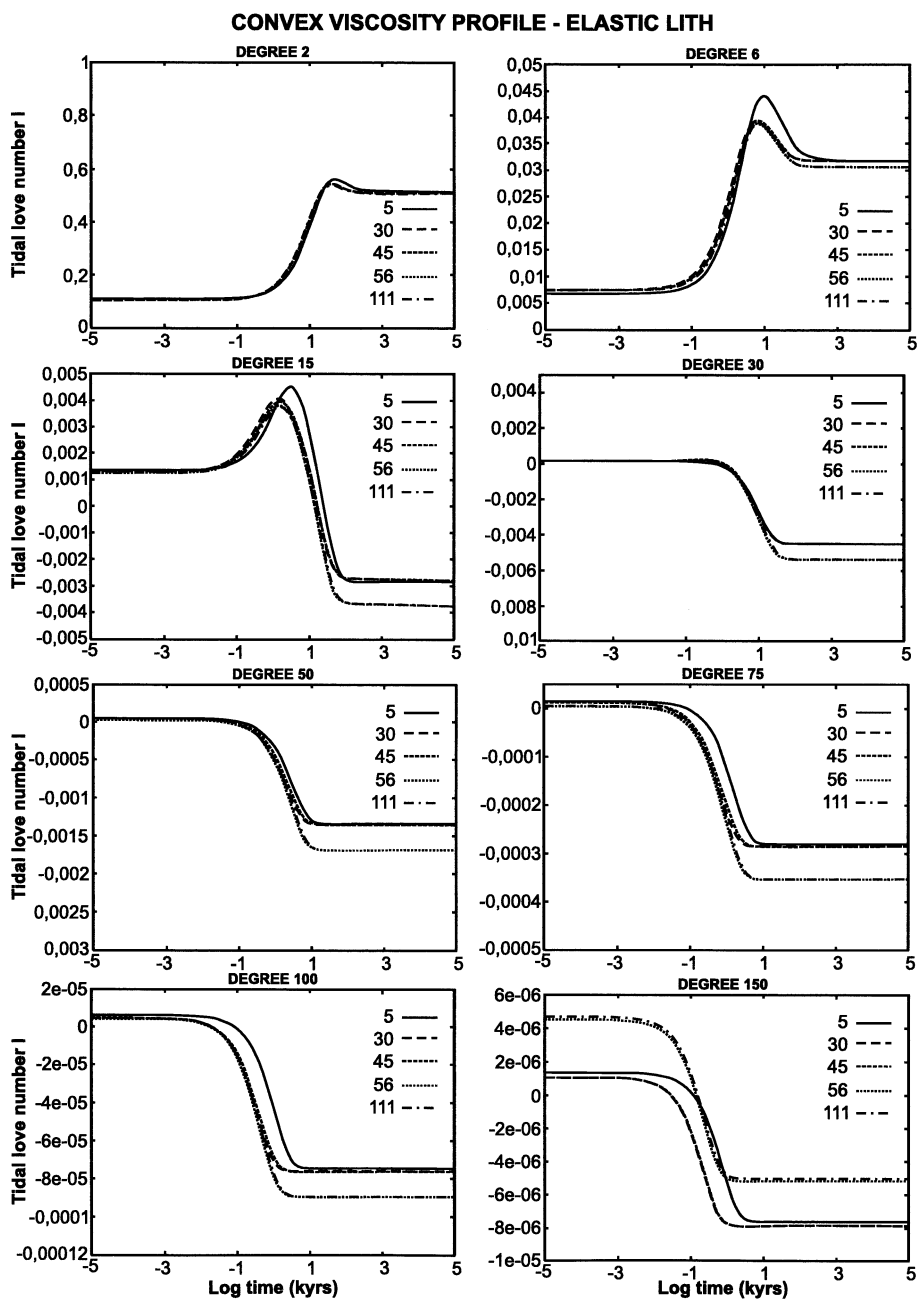


Figure 2.5c. (Continued)

1.3 CONVEX MANTLE VISCOSITY AND VISCOELASTIC LITHOSPHERE

We now show figures for the load Love number k , but will only discuss the other 5 cases (load Love numbers h and l , and tidal Love numbers k , h and l) without explicitly showing the plots as was done previously.

Figure 2.6 shows the results of the same simulations as in the prior section for the load Love number k , but now with the elastic lithosphere replaced by a viscoelastic layer with a viscosity of 10^{21} Pa s.

It is clear that for all harmonic degrees the 5-layer cases attain the expected fluid limit of -1 with a very high accuracy. Also the 30- and 45-layer cases do not show noticeable deviations, but the 56- and 111-layer cases do. The behavior of the 56-layer and 111-layer models is roughly in agreement with the fact that deviations from -1 in the fluid limit should be higher for high harmonic degrees than for lower harmonic degrees if the deviations are due to shallow layer density inversions: for low harmonic degrees the total strength is distributed throughout the whole mantle, while for high harmonic degrees the total strength is distributed mainly through the shallow mantle layers. The 30- and 45-layer models pose more problems, as the fluid limit of -1 is reached for all harmonic degrees. Simulations on the load Love number h also show that the fluid limits for the load Love number h of all 5-, 30- and 45-layer cases virtually cover each other. This looks at odds with the results in Figure 2.4a, where the fluid limits of the 5-layer models differ slightly from the fluid limits of the 30- and 45-layer cases.

In all models of the load Love number h the (absolute value of the) fluid limit should be in agreement with the notion (equation (52a) of Wu and Peltier, 1982) that for large harmonic degrees the fluid Love number gets progressively closer to the limit

$$-\frac{2l+1}{3} \frac{\rho_{av}}{\rho(a)}, \quad (2.4)$$

with l the harmonic degree, ρ_{av} the volume-averaged value for the density of the Earth model and $\rho(a)$ the density of the uppermost layer. In the cases of a uniform lithosphere density (the cases with 5, 30 and 45 layers) the fluid limits for the load Love number h are virtually indistinguishable from one another, but the fluid limits for the load Love number h in the cases of 56 and 111 layers are markedly increased in absolute value, as was the case in the models of the preceding sections. The density reduces from 3184 kg/m^3 for the homogeneous lithosphere to 1020 kg/m^3 for the uppermost layer in the 111-layer model, while the mean densities of the models remain roughly the same. This would result in an increase of the fluid limit by a factor of about 3 for constant harmonic degree. We find that the differences between the fluid limits for the load Love number h in the 5-, 30- and 45-layer cases on one hand and the 56- and 111-layers case

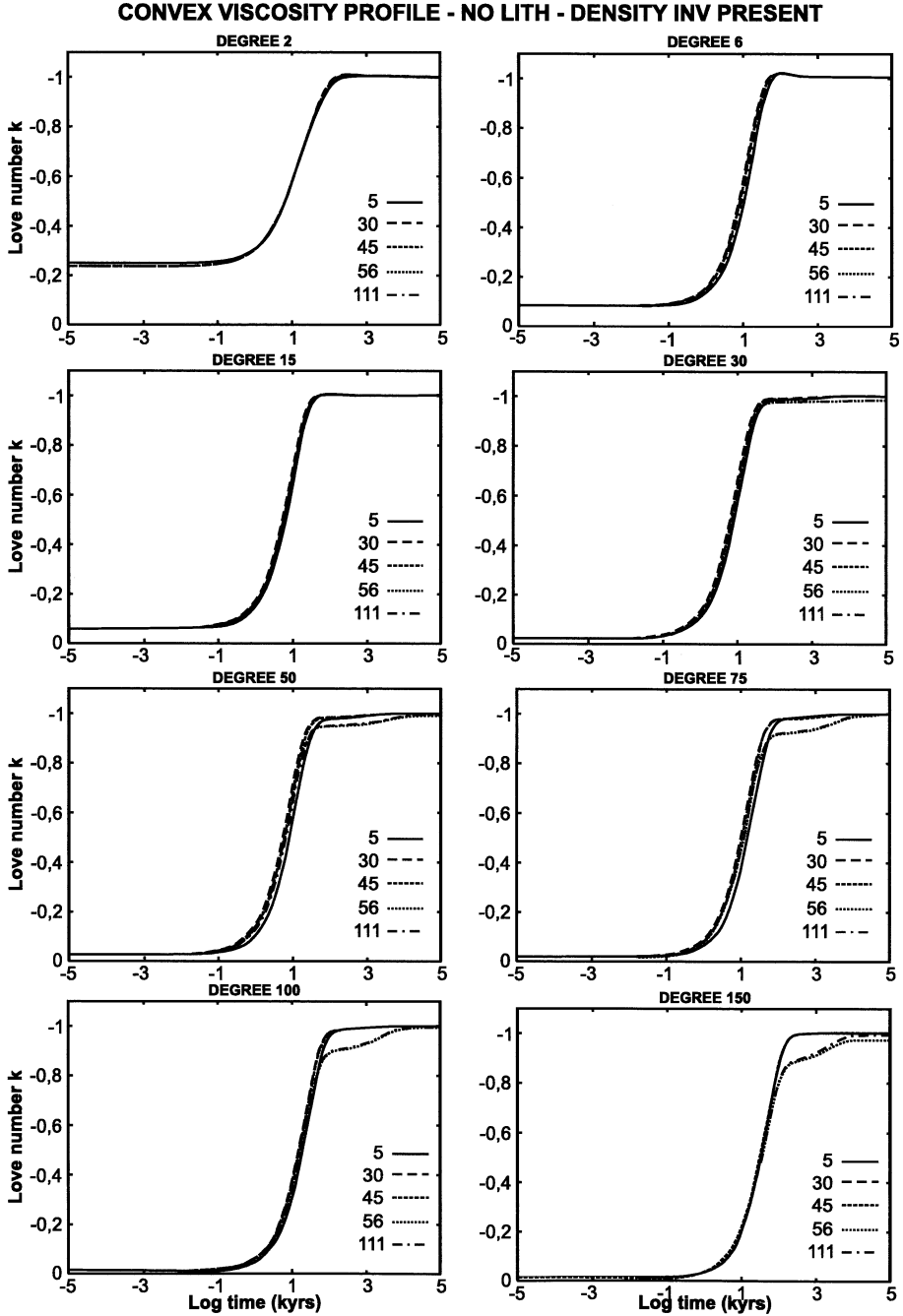


Figure 2.6. As in Figure 2.4a, except for a lithosphere with a viscosity of 10^{21} Pa s. The density stratification is again derived from PREM, which implies that the 30-, 45-, 56- and 111-layer models contain some density inversions in the lithosphere and shallow upper mantle (PREM contains density inversions between 24.4 and 220 km depth.)

on the other grow progressively with increasing harmonic degree, but remain within a factor of 2. The likely cause of this is that even for degree 150 the fluid limit is still susceptible to the ‘somewhat deeper’ layers. ‘Somewhat deeper’ means here only 3 km, as at 3 km depth the density in PREM jumps from 1020 kg/m^3 to 2600 kg/m^3 , so this might well be the cause.

The most striking feature in our simulations on the load Love number l are the enormous differences between the curves with and without an internal lithospheric layering. Apparently, when the upper part of the model does not consist of an elastic lithosphere but of a viscoelastic stratification, the load Love number l becomes extremely sensitive to the uppermost model stratification. This behavior is not restricted to high harmonic degrees: for degree 2 the differences already exceed 100 per cent.

Our simulations of the tidal forcing show quite different tendencies from the corresponding curves for the loading case. The behavior of the tidal Love number h looks remarkably like the behavior of the load Love number k in Figure 2.6. The tidal Love number k again shows a tendency that the 5-layer model has reached a somewhat different fluid limit from the other layer models for low harmonic degrees, and that the internal stratification of the viscoelastic lithosphere is of importance for the higher harmonic degrees. The tidal Love numbers l are remarkably less susceptible to the internal stratification of the viscoelastic lithosphere than the corresponding cases for the load Love numbers l for low harmonic degrees. We thus find that load relaxation is much more discriminating for shallow structure than tidal relaxation whenever the lithosphere is not purely elastic.

The conclusion from this section concerning the effects of density inversions is that they indeed might effect the results to some extent. For the high harmonic degrees the fluid limits are clearly not reached for the 56- and 111-layer models, while for the 30- and 45-layer models the situation remains unclear. Therefore, it is necessary to see what happens with the fluid limits in our models when the density inversions are deleted from the stratification. This will be done in the next section. Whenever the fluid limits for the load Love number k deviate from -1 , it will be an indication that the accuracy of the model is not high enough; whenever the fluid limits become equal to -1 , it will be clear that in those cases the missing, unstable buoyancy modes are the cause. In the next section it will thus become clear whether the discrepancies which followed from the simulations in this section are due to modeling inaccuracies or to density inversions.

2. SHALLOW INVERSIONS IN PREM DENSITY STRATIFICATION

PREM contains density inversions between 24.4 km depth and 220 km depth (Dziewonski and Anderson, 1981). In this interval of 195.6 km thickness, the

density reduces from top to bottom, from 3380.76 kg/m^3 to 3359.50 kg/m^3 . Although this reduction of 21.26 kg/m^3 , i.e. less than 1 per cent, is quite small, it can lead to computational problems as such density inversions can become prone to Rayleigh-Taylor instabilities. Although most modelers seem to have been well aware of this, Plag and Jüttner (1995) have given a more explicit assessment of the role played by these density inversions in normal mode relaxation models. Most of the layers in which the density inversions occur are situated within the lithosphere, taken to be 120-km thick in our models. In those lithospheric layers, if the rheology at least is elastic, Rayleigh-Taylor instabilities do not occur. In the viscoelastic layers between 120 km and 220 km depth, the buoyancy modes will be associated with positive roots of the secular determinant, implying that the contributions from these modes will in principle show an exponential growth instead of an exponential decay. Strictly speaking, as Plag and Jüttner (1995) point out, the linearized normal mode method breaks down after some time as the deformation associated with these Rayleigh-Taylor unstable modes can become too large.

In the preceding sections we have simply neglected the buoyancy modes arising from density inversions, i.e. we have not taken any contributions into account from modes with a positive root. In this section we investigate whether neglecting these contributions leads to significantly different curves for the Love numbers if compared with a model in which the layers between 24.4 km and 220 km depth have the uniform volume-averaged density of these layers. We are thus concerned here only with the practical point of view regarding the accuracy of the relaxation models, not with a fundamental discussion of these shallow mantle density inversions and their consequences (which inevitably would have to include topics like lateral variations and mantle convection). It should be noted that this problem of shallow density inversions occurs in all relaxation models and is not unique to a normal mode model.

The best way to test our models is to use a convex viscosity profile of the mantle and a viscoelastic lithosphere with a viscosity of 10^{21} Pa s . Not only do we have inverse density contrasts on all the layers between 24.4 km and 220 km depth, which lead to unstable modes (in contrast to an elastic lithosphere), but we also know beforehand that the load Love numbers k in the fluid limit should be -1 for low harmonic degrees in the case of the absence of density inversions. For large harmonic degrees there might be a caveat worth considering. If a load is positioned at the top of the surface, hydrostatic equilibrium will be reached after some time has elapsed. But when this condition of hydrostatic equilibrium is reached, the load will still be on top of the surface. The surface is deflected in such a way that hydrostatic equilibrium is reached, implying that the density distribution of the Earth immediately beneath the surface is now somewhat further away from the geoid than the load. This leads locally to a slight deviation from the fluid value of -1 . It is clear that only for high

harmonic degrees - if detectable at all - some small deviations for the load Love number k might be expected (see also the discussion in section 7.1 on a possible overcompensation in the topography of the Himalayas by its root). In the ideal case of a unit mass with no constraints on its dimensions, this problem does not play a role. So if we find any deviations from the fluid limit for models which do not contain inversions in the density profile, then we are virtually sure that these deviations are not due to density inversions but to inaccuracies in the calculations. Besides the load Love number k fluid limit of -1 , we can expect beforehand that inaccuracies in the calculations will become most pronounced in cases in which the lithosphere is absent, as the calculations break down towards the $s = 0$ (fluid) limit (Wu and Peltier, 1982). Models which have an elastic lithosphere can thus in general be expected to show smaller calculation inaccuracies and smaller differences between models which do have shallow density inversions and corresponding models which do not have them.

Figure 2.7 shows that for harmonic degrees up to 75 all curves of the load Love number k reach the fluid limit of -1 with high accuracies. For the degrees 100 and 150 this is also the case for the 5-, 30- and 45-layer cases, but the 56- and 111-layer cases deviate from the ideal limit. Apparently, the accuracy breaks down in our models for high harmonic degrees (higher than 100) when the number of layers is larger than about 50. This accuracy limit can be understood as a collapse of the fundamental matrix for s -values very close to zero (the matrix becomes singular then as the matrix becomes over-determined for a pure fluid rheology), and a shift of the roots of the secular determinant to smaller s -values when the harmonic degree increases. When we compare the results in Figure 2.7 with the corresponding cases in Figure 2.6, we see that there is some influence of the inversed density layers for high harmonic degrees and fine-layered stratification.

In our simulations for the load Love number h , a remarkable phenomenon becomes apparent when we compare the load Love numbers h resulting from the models with and without the density inversions. For all harmonic degrees the fluid limits of the 5-, 30- and 45-layer cases as a function of harmonic degree are the same between the two cases, but the fluid limits for the low harmonic degrees in the 56- and 111-layer cases are significantly larger in the case in which the density inversions are deleted than in the corresponding cases in which the density inversions are retained. For harmonic degree 75 the fluid limits have virtually become the same again, however, while for degrees 100 and 150 the effect of the numerical accuracy becomes operative. However, the degrees 100 and 150 show fluid limits for the 56- and 111-layer cases in the models where the density inversions are deleted which are much closer to the fluid limits of the corresponding models in which the density inversions are retained than for the low degrees. So it looks like this remarkable behavior of the low-degree fluid limits for the load Love number h is a physical effect. This

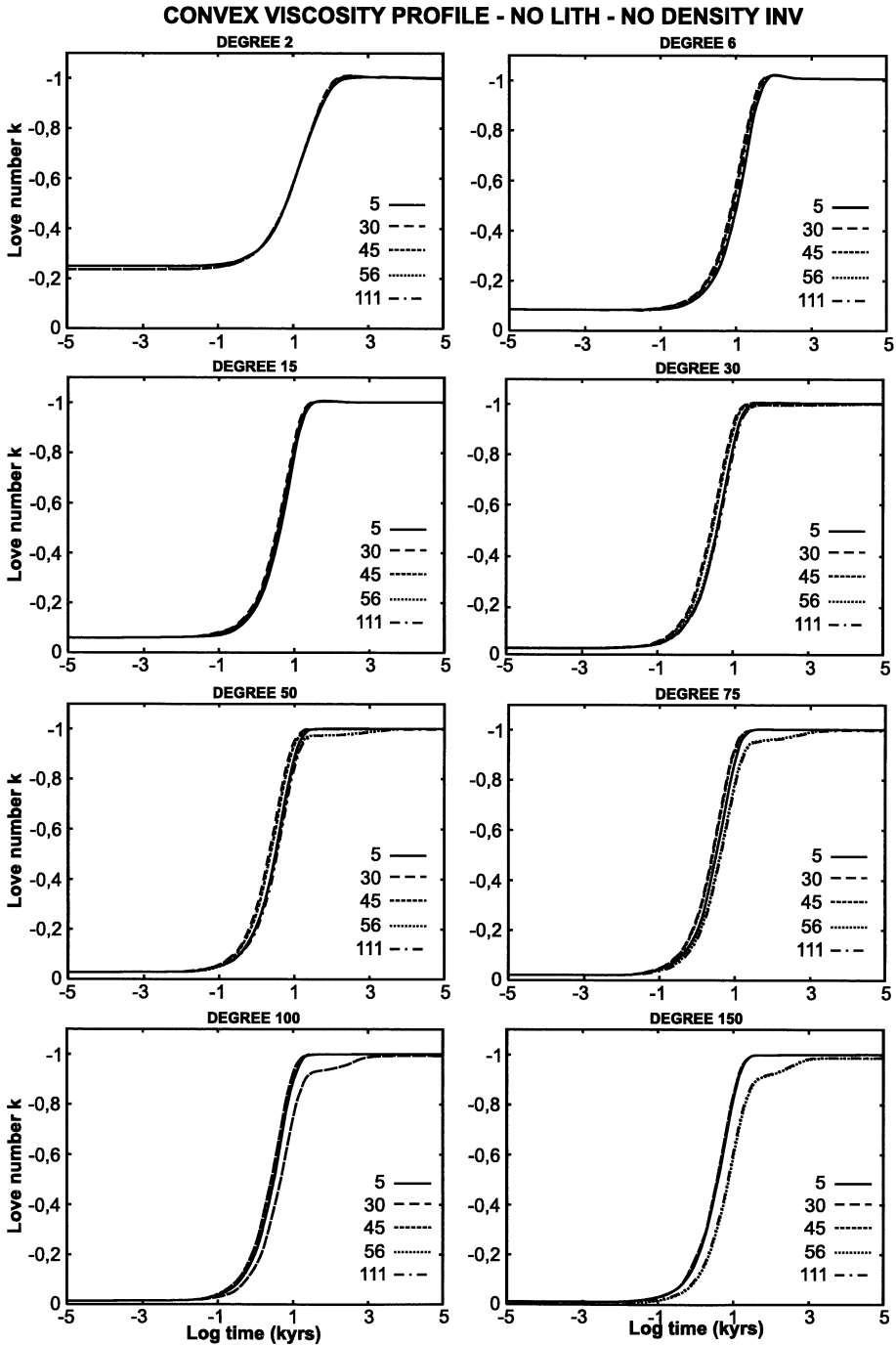


Figure 2.7. As in Figure 2.6, but with the density inversions deleted. The region between 24.4 and 220 km depth is homogeneous with respect to the density (but not with respect to the rigidity and viscosity).

behavior was also visible, and even stronger, in the curves of the load Love number l .

From the comparison of the models in which the shallow density inversion was deleted or retained, it follows that especially for the load Love numbers h and l the influence of the shallow density inversions is very strong whenever the lithosphere is not elastic. This influence is not confined to large harmonic degrees, but is even stronger for low harmonic degrees whenever the viscoelastic lithosphere layers are stratified (recall that the lithosphere contains most of the layers which have density inversions in PREM).

Whereas the differences for the load Love numbers were found to be dramatic for h and l , differences between the PREM-stratification and the stratification without density inversions become much reduced and are often not discernible anymore in the case of tidal forcing. Especially the tidal Love numbers k and h showed the same curves in almost all cases between the deleted and retained density inversion models. Only the tidal Love number l simulations showed differences, but even here the differences never became so large for the 56- and 111-layer cases as for the load Love number cases. One can thus conclude from these simulations that the effects of inversions in density stratification depend heavily on whether the forcing loads the surface of the Earth (as is the case with ice sheets) or not (tidal interaction).

In general, the results from this section show clearly that the differences in fluid limits in the preceding section are due to the density inversions in stratification which PREM contains. In reality, these density inversions will most likely cause less trouble in modeling, as the lithosphere is usually assumed to be elastic. It makes sense that these density inversions cause such great discrepancies whenever the lithosphere is viscoelastic, as layers which contain density inversions and which have low values for the viscosity (in our case 10^{21} Pa s) will easily become unstable by Rayleigh-Taylor instabilities and will readily start to convect, thus rendering the normal mode formalism inapplicable after a relatively short time.

3. VOLUME-AVERAGED VS. FIXED-BOUNDARY CONTRAST

The 5-layer models described previously had densities which were determined by volume-averaging the densities of the PREM stratification. In the past, simple-layer models (less than or equal to 5 layers) have usually been built in such a way that the density contrasts at the various internal interfaces were kept the same as in PREM. From the results of the previous sections it can be deduced that volume-averaging gives results which do differ from the corresponding models with larger numbers of layers, but these differences are usually not dramatic. It is thus very interesting to see how fixed-boundary contrast models will behave compared to multi-layer PREM.

The motivation for building 5-layer volume-averaged models instead of fixed-boundary contrast models is that in a crudely-layered model relaxation is more likely to feel crude mean values for the constitutional and rheological parameters than to experience subtle density contrasts at the various internal boundaries. Or to put it differently: the relaxation will also experience the density increases or decreases further away from an internal boundary, not only at the internal boundary. A second degree zonal harmonic degree term will not notice a density contrast over a few kilometers at 670 km depth; it will at most only notice a crude upper-lower mantle density contrast.

If one compares the density contrasts at the lithosphere-shallow mantle boundary, the shallow mantle-transition zone boundary and the transition zone-lower mantle boundary in Tables 2.1 and 2.2, then one finds that the density contrasts at these three boundaries are considerably higher in the volume-averaged cases than in the corresponding fixed-boundary contrast cases (with even no density jump at all at 120 km depth in the fixed boundary contrast case). But the absolute values for the densities of the lithosphere, the uppermost mantle and the transition zone are, as a consequence, considerably larger in the fixed-boundary contrast models. This might give a trade-off between a decrease in the boundary contrast and a corresponding increase in the absolute value of the density for some geophysical signatures.

Another related point is the following: as was shown in the former sections, the relaxation of a multi-layer PREM consists, apart from the elastic response, of the total contribution from the buoyancy modes and the viscoelastic modes (that is, for an incompressible model; for a compressible model the viscoelastic layers trigger extra denumerably infinite sets of dilatation modes). The strength of the buoyancy modes goes to zero when the density contrast goes to zero, but not the strength of the viscoelastic modes. This implies that in a multi-layer model with a large amount of discrete layers one ends up with the contributions from those layers which have finite density jumps, supplemented with the viscoelastic modes. In fact, all the layers in, say, a 56-layer model have boundary contrasts, but the buoyancy contributions from all these jumps is close to zero, except for a few (notably those at the CMB, at the 670 km discontinuity and at the lithosphere-uppermost mantle interface). It is thus of great importance not to confuse a 5-layer fixed-boundary contrast model with PREM in which only the contributions of the relaxation of the surface, CMB, 670 km discontinuity, 420 km discontinuity and the lithosphere-uppermost mantle interface are taken into account: although the density contrasts at the various boundaries are the same, the absolute values of the densities in the 5-layer fixed-boundary contrast model differ from the mean values of the densities of the corresponding layers in PREM.

Figure 2.8 portrays the comparison between the 5-layer volume-averaged models and the 5-layer fixed boundary-contrast model for the Love number k :

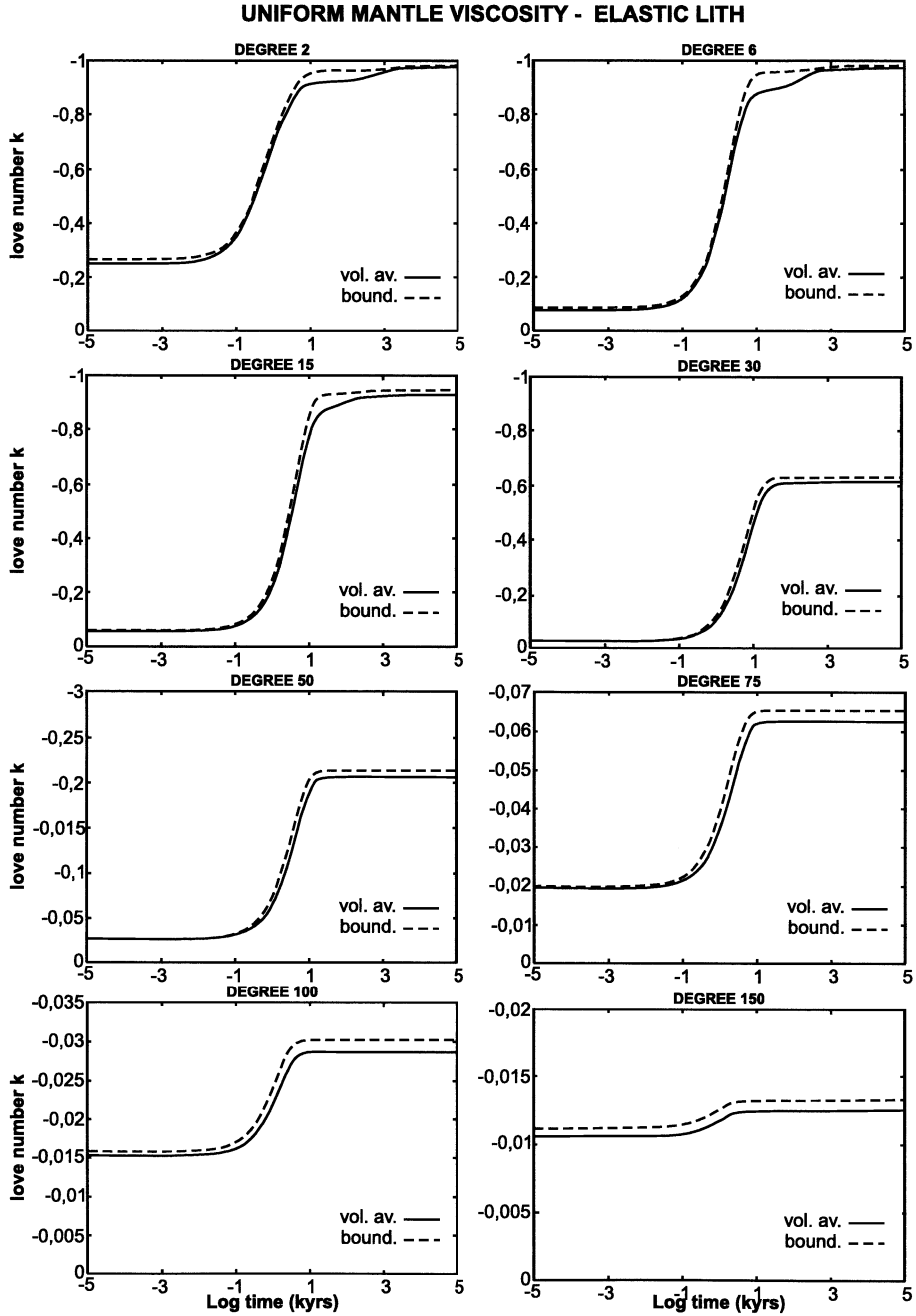


Figure 2.8. Comparison between the 5-layer volume-averaged models and 5-layer fixed boundary-contrast models. The 5-layer volume-averaged models are exactly the same as those used in Figure 2.1a. For the description of the various panels see the caption to Figure 2.1a. The stratification and values for the densities and rigidities of both volume-averaged and fixed boundary-contrast 5-layer models are given in Tables 2.1 and 2.2.

and surface load case. The mantle has a uniform viscosity of 10^{21} Pa s, so the 5-layer volume-averaged cases are the same as those depicted in Figure 2.1a. It was shown that the 5-layer cases are not very different from the cases in which the stratification has a larger number of layers, especially for low degree harmonics. Figure 2.8 shows that the differences between the two ways of modeling become most substantial at transient timescales for low harmonic degrees. The volume-averaged models systematically result in lower absolute values for the load Love number k on all timescales compared with the fixed-boundary contrast models, indicating that for a fixed time the relaxing medium in volume-averaged models is further from hydrostatic equilibrium than the fixed-boundary models on all timescales. At first sight the differences might not seem dramatic, but in comparing the results between the two modeling approaches one should rather determine the differences with respect to $k = -1$ than to $k = 0$, and then the differences become substantial. The differences between the volume-averaged model and the fixed-boundary model are seen to become independent of time for high harmonic degrees. This trend was also visible in our simulations for the case of the load Love number h , but not for the case of the load Love number l . For low harmonic degrees the curves for the h Love numbers showed that differences between the two model approaches only become significant on long timescales, and that these differences are larger for fixed harmonic degree than in the case of the k Love number. This behavior also occurs in the case of the load Love numbers l for degree 2 and for large harmonic degrees. For intermediate harmonic degrees, differences on all timescales occur, while at degree 15 there is a transition from l -values becoming larger in the fixed-boundary case to l -values becoming larger in the volume-averaged case. This transition is accompanied by a sign change of the fluid limit.

In general, the differences between the volume-averaged and fixed-boundary value models for the h and l Love numbers in the case of tidal forcing were observed to be smaller than in the case of a surface load. On the other hand, these changes are larger in the case of the tidal Love number k . In the case of the tidal Love number l the curves were found to cross each other for all harmonic degrees, whereas for the load Love number l this happened only for degree 15.

From these models it can be concluded that 5-layer models can exhibit considerable differences when the stratification is determined by means of a volume-averaged procedure or by means of the fixed-boundary contrast method. The results in the previous sections suggest that the results from 5-layer volume-averaged models are closer to PREM-models than the 5-layer fixed-boundary contrast models, so this would indicate that the volume-averaged models are preferable if one is using models with only 5 layers.

4. ISOLATION FUNCTIONS

Fang and Hager (1995) have coined the term ‘isolation function’, which for a discrete mode structure is a step-like function of residues as a function of the inverse-time variable s . Each step j in this function occurs when s becomes equal to a pole s_j . This concept plays an important role in their ‘continuous poles’ treatment of performing the Laplace inversion, and it is thus of interest to see whether this ‘isolation function’ concept shows the ‘failure of the discrete mode structure’ for continuous viscosity profiles, as they claim on page 858 of their paper (Fang and Hager, 1995, bottom of right column).

Fang and Hager (1995, page 855, top of left column) define the isolation function I_l in the case of M poles as

$$I_l(s, t) = \sum_{j=1}^M \mathbf{K}_{lj} e^{s_j t} H(s - s_j), \quad (2.5)$$

in which $\mathbf{K}_{lj}$ is the residue of the pole defined in equation (1.137) and H the Heaviside function. I_l is evaluated for a fixed time t . In order to compare some of our results with Fang and Hager’s (1995) results, we have determined the isolation functions of the linear viscosity profiles of sets A and B in their Figure 7. Note that the models in their Figure 7 show an inverse kind of viscosity profile as a function of depth with respect to the viscosity profile in our Figure 2.3. We have taken the same model parameters as Fang and Hager (1995), except for the lithosphere in set A. Fang and Hager (1995) state in the caption to their Figure 7 that they have taken an exponentially increasing viscosity in the lithosphere of their set A, ranging from 10^{21} Pa s at the base (120 km depth) to 10^{26} Pa s at the surface. We have taken a completely elastic lithosphere instead. We plot the Love numbers on the vertical axes rather than the residues as Fang and Hager do. This difference is not of any importance in comparing results, as the Love numbers are merely a renormalization of the residues by a constant factor. After all, our interest in this section is whether the remarkable continuous/discontinuous behavior of Fang and Hager’s (1995) Figures 8 and 9 shows up in our simulations.

Our modeling results for the rheological model consisting of an elastic lithosphere on top of a homogeneous upper mantle with a viscosity of 10^{21} Pa s and a linear viscosity increase in the lower mantle are shown in Figure 2.9.

The plots clearly show a transition from curves with only discrete jumps to curves which contain both a continuous behavior in combination with discrete jumps when the number of layers is increased. The continuous behavior is associated with the viscoelastic modes, triggered by the viscosity increase in the lower mantle. It might seem strange at first sight that these viscoelastic modes are occurring on long timescales, as they occur on short timescales (Han and Wahr, 1995), but in the simulations of Figure 2.9 there are no density contrasts

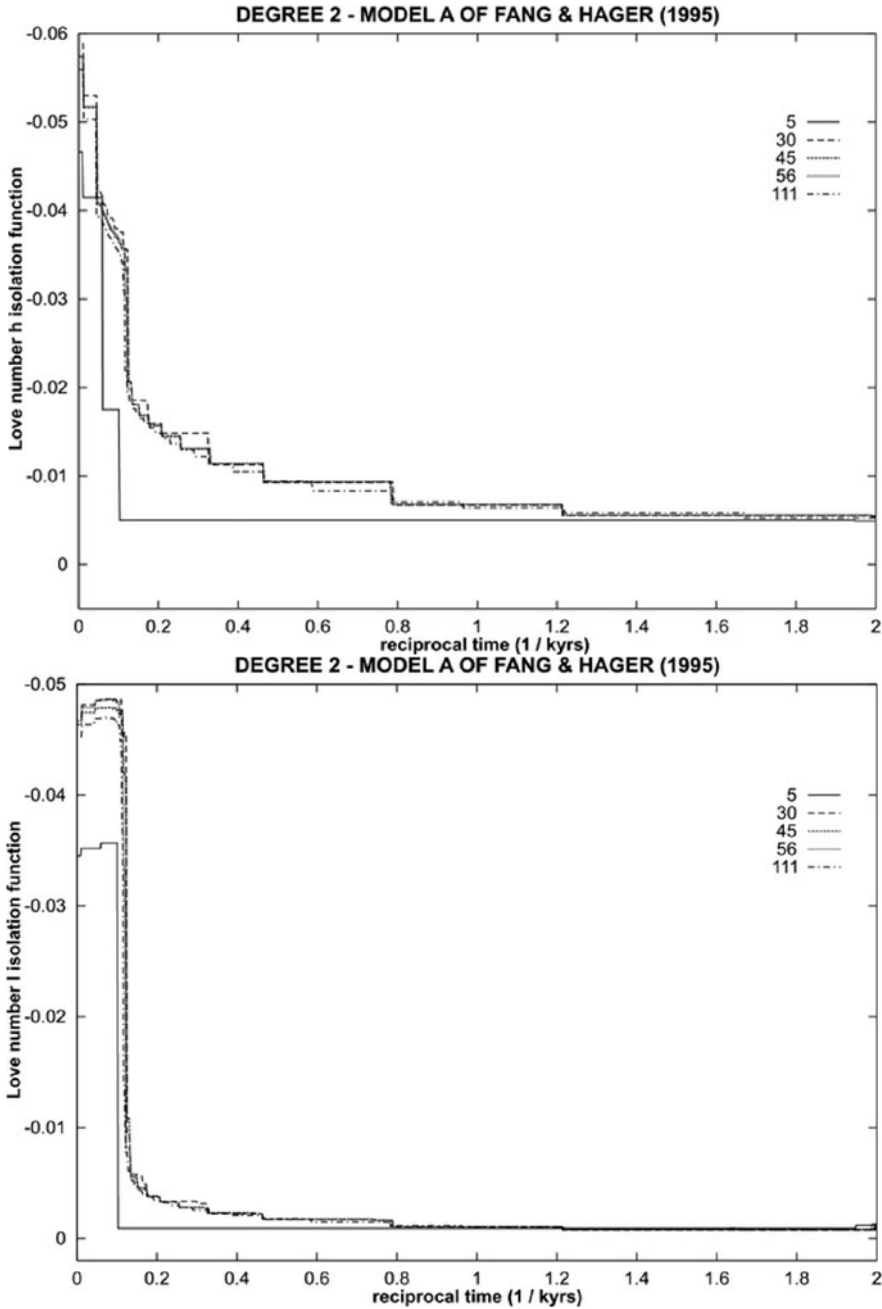


Figure 2.9a. Isolation functions for the linear upper mantle viscosity profile of case A in Figure 7 of Fang and Hager (1995). The figure shows the isolation function for (a) the degree 2 h and l Love numbers as a function of reciprocal time for the 5-, 30-, 45-, 56-, and 111-layer cases, (b) the degree 4 h and l Love numbers, and (c) the degree 15 h and l Love number. In panels (a) and (b) the isolation functions are plotted at time 1.2 kyr; in panel (c) at time 4.2 kyr. Redrawn from Figure 9 in Vermeersen and Sabadini (1997).

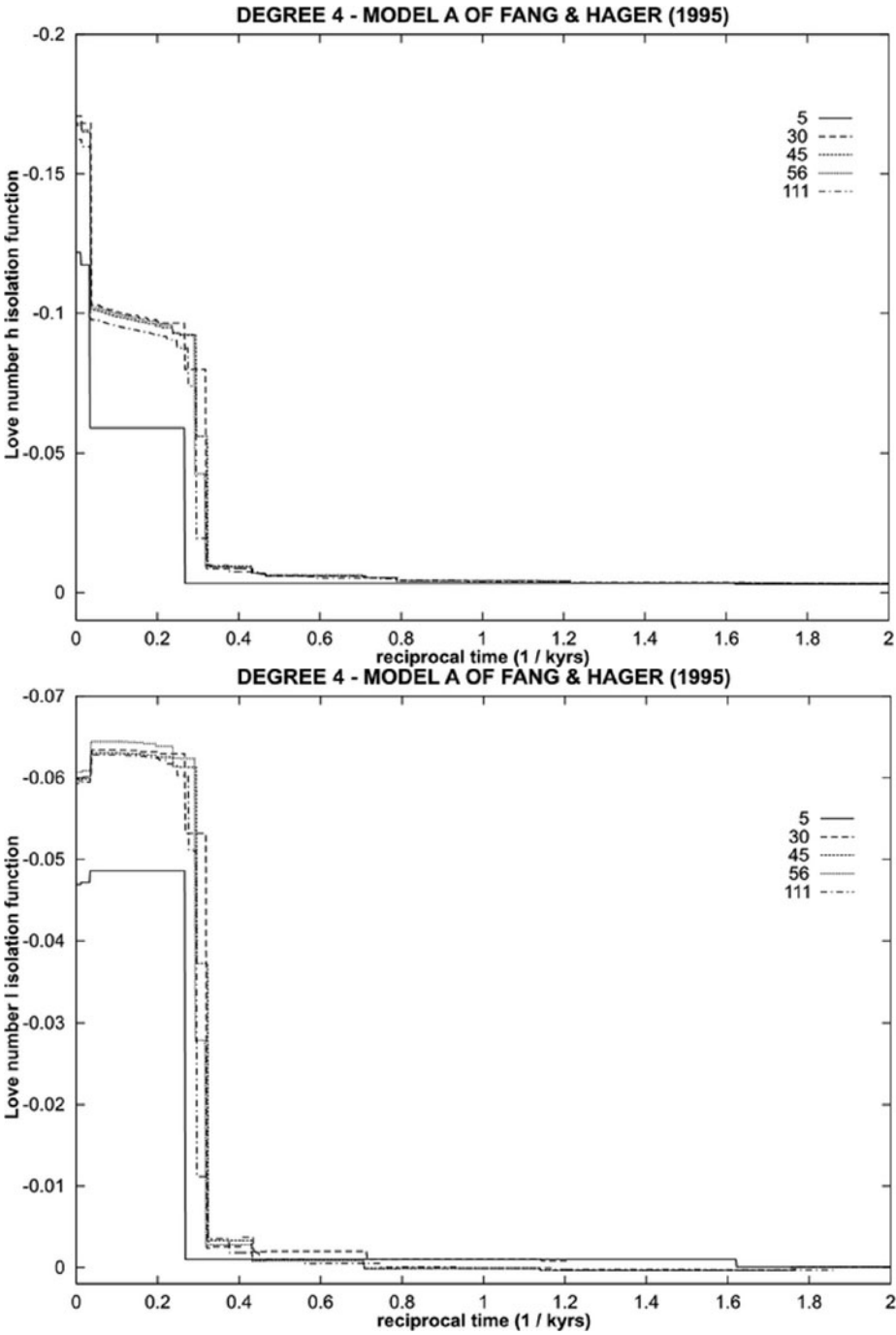


Figure 2.9b. (Continued)

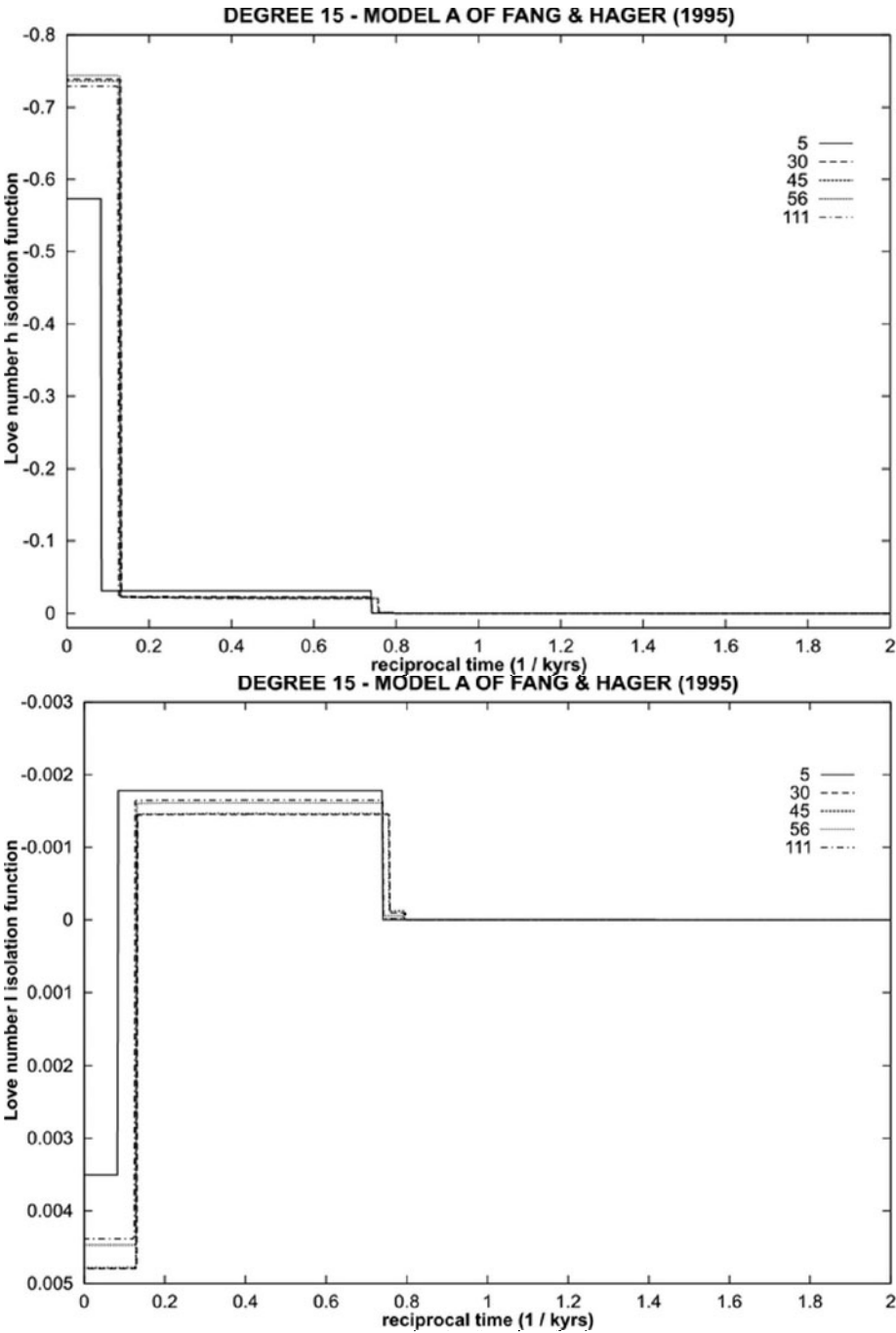


Figure 2.9c. (Continued)

inside the lower mantle. When the boundary between two viscoelastic layers does not contain a jump in density (thus not triggering a buoyancy mode), the two viscoelastic modes that are triggered appear to have long relaxation times. So, when the density contrast between two layers is reduced to zero, the relaxation times of the viscoelastic mode pair progressively become larger. In all the panels of Figure 2.9, it might seem that on longer timescales (the left side of the curves) the continuous behavior is more pronounced than on shorter timescales (the right side of the curves). The reader should be aware, however, that the scale of the horizontal axis is non-linear in time (the s -scale is linear, but not the $1/s$ -scale): the longer timescales are strongly compressed.

When we compare our results in Figure 2.9 for harmonic degree 4 with the result of Fang and Hager's (1995) Figure 8 more closely, we see a number of similarities, but also some differences. In the comparison only the fine solid lines in the Fang and Hager's (1995) Figures 8 and 9 are of importance (depicting their linear viscous profile cases). Their thick solid lines should not be confused with our 5-layer cases: although our 5-layer cases have a homogeneous lower mantle, we take volume-averaged values of the linear viscosity profile, which clearly results in another lower mantle viscosity than Fang and Hager's (1995) set A in their Figure 7. The relatively large jump in our Figure 2.9b for the h Love number at about 0.3 1/kyr is also present in the first panel from the top of Fang and Hager's (1995) Figure 8, but for longer timescales our curves show a combination of a smooth continuous increase, followed by a strong discrete step at about 0.05 1/kyr. Their curve (the fine solid one) shows a more discrete step-like behavior. Our Figure 2.9b for the l Love number looks somewhat more like the curve of the second panel in Fang and Hager's (1995) Figure 8 in this respect. Our Figure 2.9c for harmonic degree 15 shows no continuous behavior anymore, and in this respect our curves look much like the fine solid line curves in the third and fourth panels of Fang and Hager's (1995) Figure 8. The largest difference occurs for the l Love number: the curves in the lower panel of our Figure 2.9c do not contain any jumps between 0.2 and 0.6 1/kyr, while the fine solid curve in the last panel of Fang and Hager's (1995) Figure 8 contains two (one unlabeled close to 0.2 1/kyr and one labeled 'V'). The curves in our Figure 2.9c show that for degree 15 the viscoelastic modes of the lower mantle do not carry much strength anymore, leaving only the two buoyancy modes present (those induced by the surface and the CMB; note again that the boundary between the lithosphere and the mantle triggers no buoyancy mode because there is no density jump). Comparing the behavior of the curves depicted in panels (a), (b) and (c) in Figure 2.9 shows a gradual decrease of the contributions of the (continuous) viscoelastic modes of the lower mantle, in agreement with the notion that the higher the harmonic degree becomes, the less sensitive it is to the lower mantle modes. This latter point should not be the case anymore with modes resulting from a linear viscosity profile in the

upper mantle instead of in the lower mantle: we would thus expect to keep on seeing continuous behavior resulting from the viscoelastic modes in the curves for harmonic degree 15 in our simulations of Fang and Hager's (1995) set B.

Figure 2.10 contains the simulations for these models, having a homogeneous lower mantle of 10^{21} Pa s and a linear increase in the viscosity from the upper mantle-lower mantle boundary towards the surface.

The curves in Figure 2.10a-e do indeed show a continuous behavior for all harmonic degrees. In order to outline the continuous character somewhat better, we have enlarged the left regions of the harmonic degree 4 plots (upper panels in (b) and (c) for the h Love number and l Love number, respectively) and plotted them in the lower panels in Figure 2.10b and Fig2.10c. These enlargements clearly show the development from a discontinuous profile for the 5-layer cases into a continuous profile for the 111-layer cases. In the lower panels in (b) and (c), the discrete steps become progressively smaller from right to left until the discreteness of the steps is hardly visible anymore for the largest timescales (smallest s -values) on which the viscoelastic modes occur. Note, again, that this progression from discreteness to continuous behavior is largely induced by the fact that the horizontal axis is non-linear in time: the plots have progressively more temporal resolution going from the left side to the right side.

Another interesting aspect emerging from the various panels in Figure 2.10 is that the strongest mode of the 5-layer model seems to be split into smaller fragments with increasing number of layers. We identify the discontinuous jump on the right side of the panels at about 1.65 per kyr in Figure 2.10a with the buoyancy mode of the surface. The discontinuous jump at about 0.15 1/kyr is associated with the buoyancy mode of the CMB (the third one from the right in the curve of the 5-layer model): as in our models this mode disappears if the density of the core is set equal to the density of the mantle. Note that the position of the buoyancy modes from the CMB and from the surface in Fang and Hager's Figure 9 is also internally inconsistent with the positions of these modes in their Figure 1: in all the models displayed in Fang and Hager's Figure 1 all the modes labeled 'C' have longer relaxation times than the modes labeled 'M'. The large jump in between the two buoyancy modes in the 5-layer model is associated with one of the viscoelastic modes of the upper mantle-lower mantle boundary. Apparently, the large viscosity jump at this boundary is responsible for triggering the strongest relaxation mode in this model. So from left to right we have for the curves of the 5-layer model the surface buoyancy mode, a viscoelastic mode associated with the upper mantle-lower mantle boundary, the buoyancy mode of the CMB and 5 viscoelastic modes (one of the viscoelastic modes has a strength which is so small that it is not visible in Figure 2.10a). In the top panel of Figure 2.10b on the degree 4 h Love number isolation function, the contribution from the surface buoyancy mode has virtually disappeared (although the very small jump is still just visible

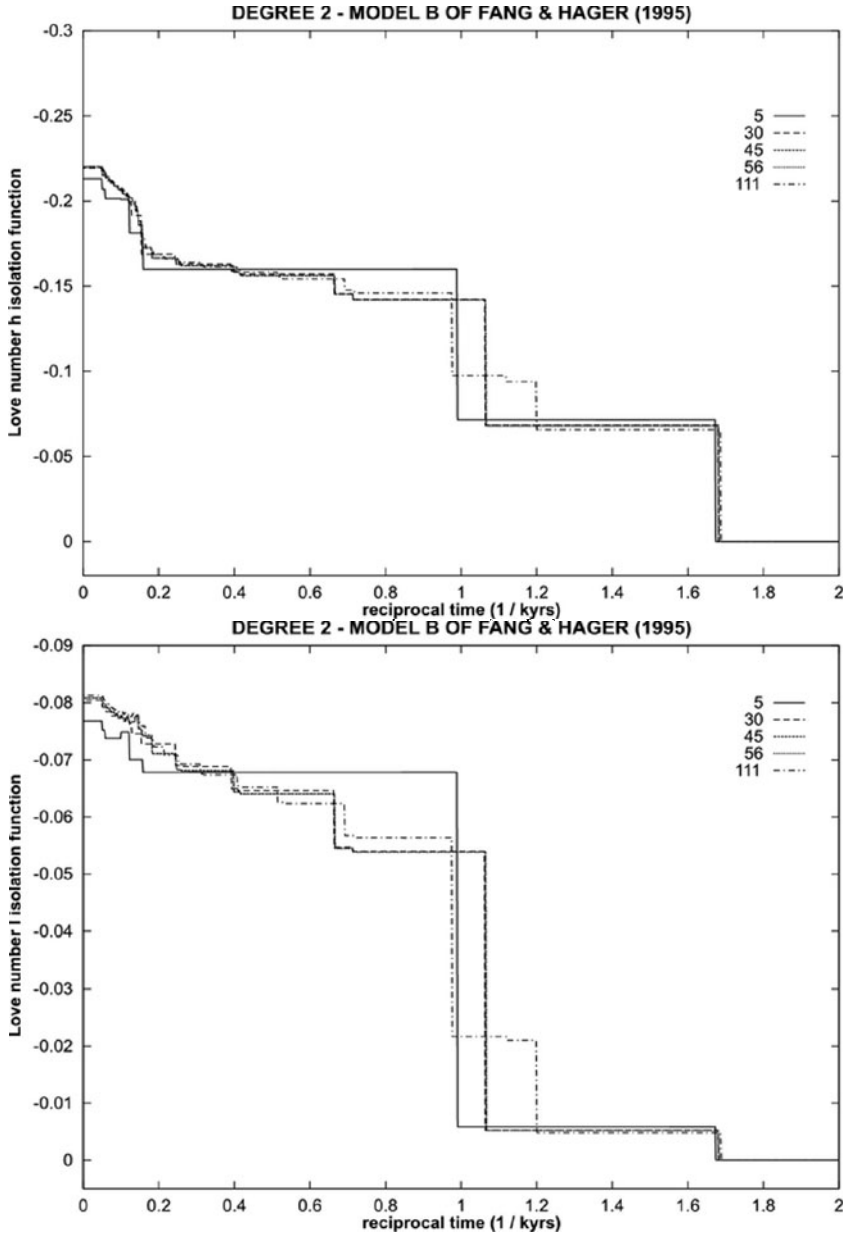


Figure 2.10a. Isolation functions for the linear upper mantle viscosity profile of case B in Figure 7 of Fang and Hager (1995). The figure shows the isolation functions for (a) the degree 2 h and l Love numbers as a function of reciprocal time for the 5-, 30-, 45-, 56- and 111-layer cases, (b) the degree 4 h Love number, (c) the degree 4 l Love number, and (d) the degree 15 h and l Love numbers. The bottom panels in (b) and (c) show enlargements of the left-hand parts of the top panels. In panels (a)-(c) the isolation functions are plotted at time 1.2 kyr; in panel (d) at time 4.2 kyr. Redrawn from Figure 10 in Vermeersen and Sabadini (1997).

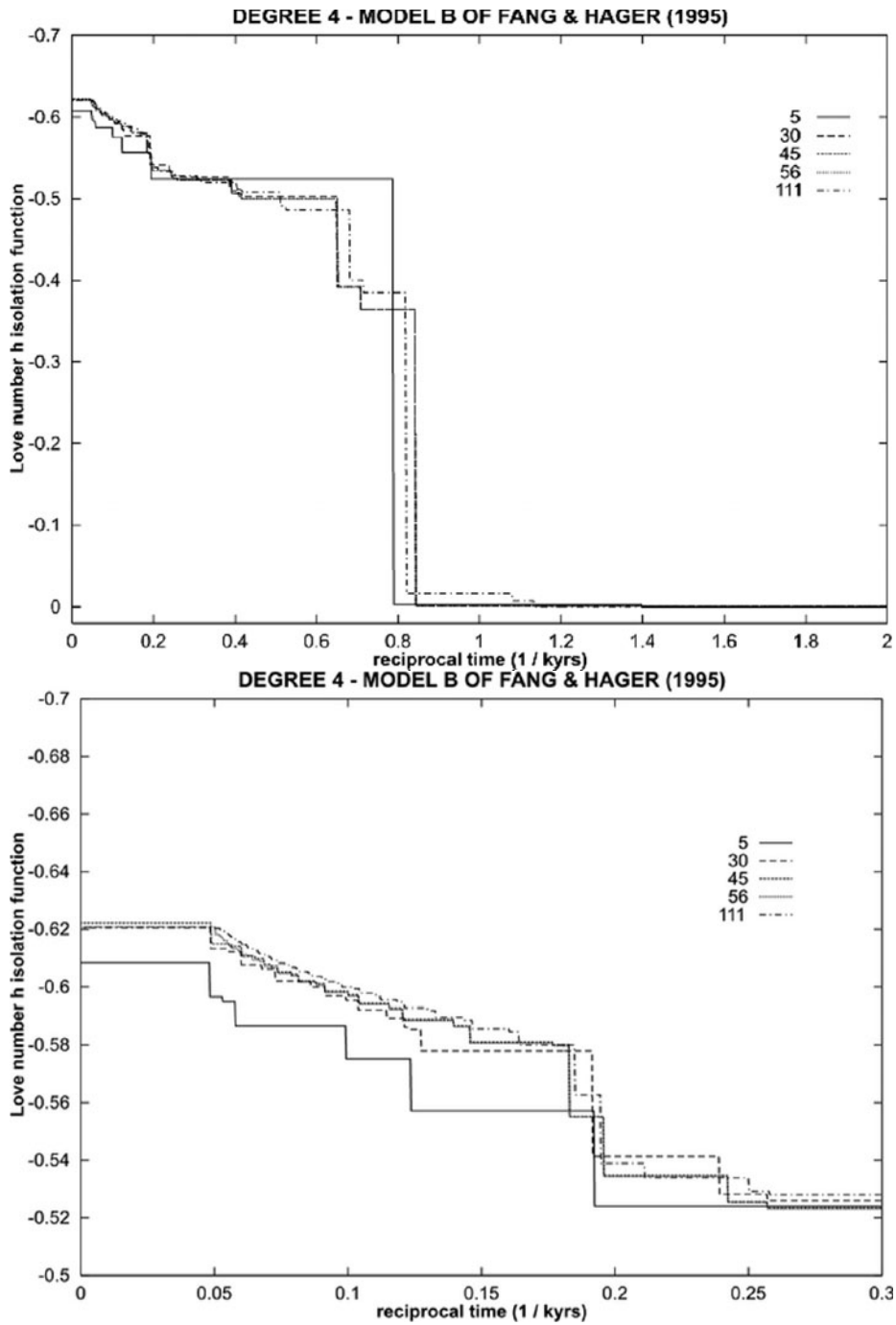


Figure 2.10b. Continued. degree 4 h Love number.

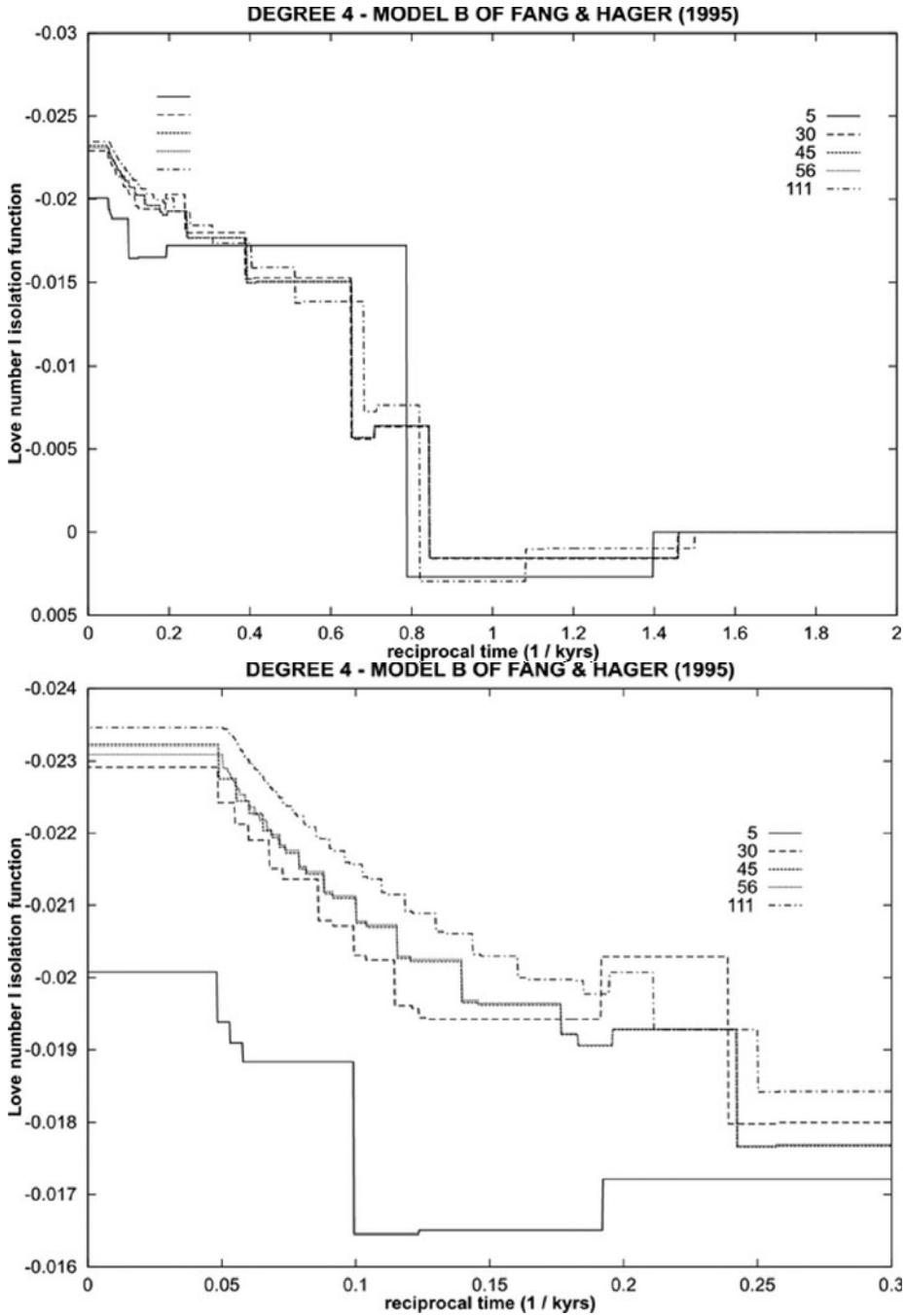


Figure 2.10c. Continued. Degree 4 l Love number.

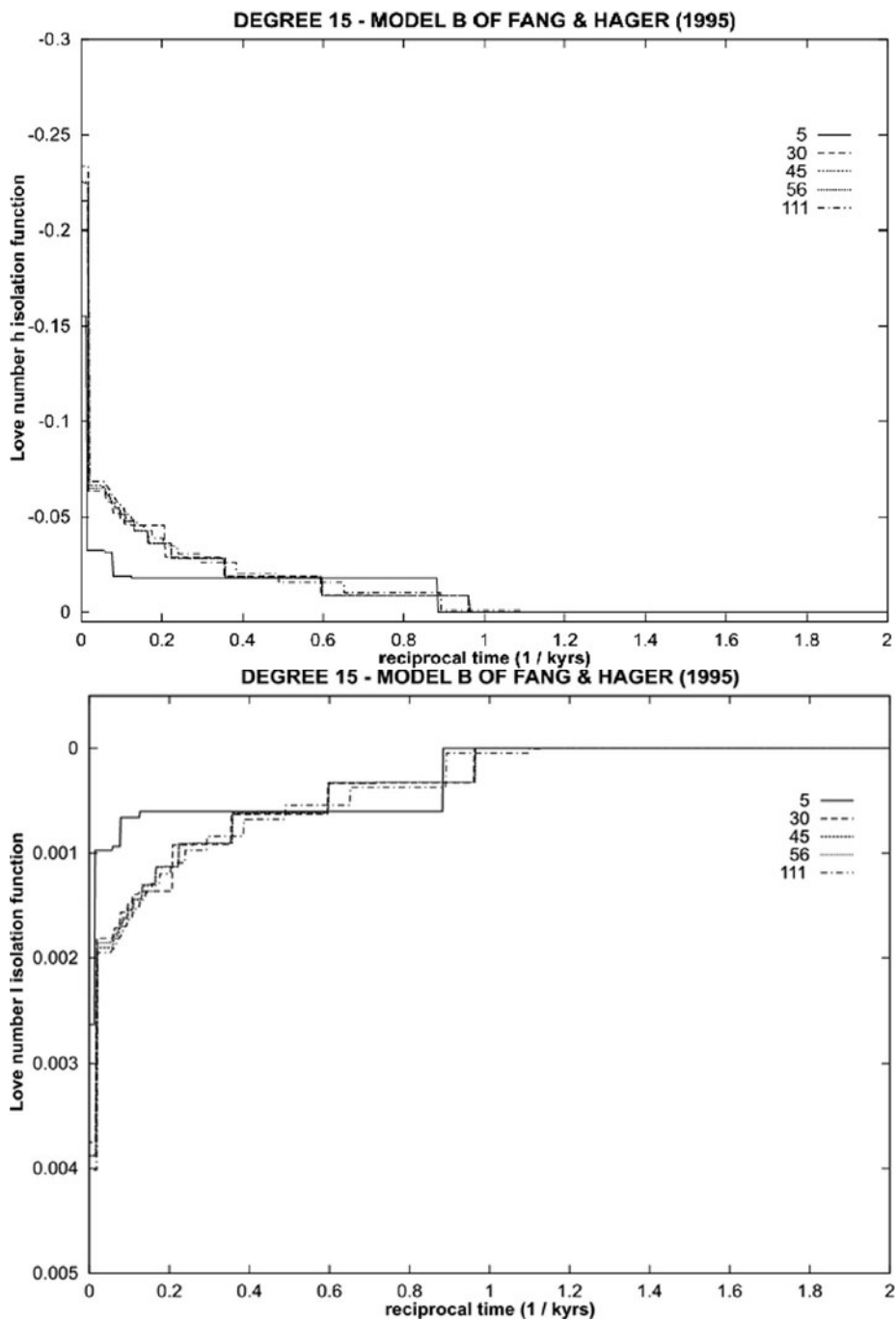


Figure 2.10d. Continued. Degree 15 h and l Love numbers.

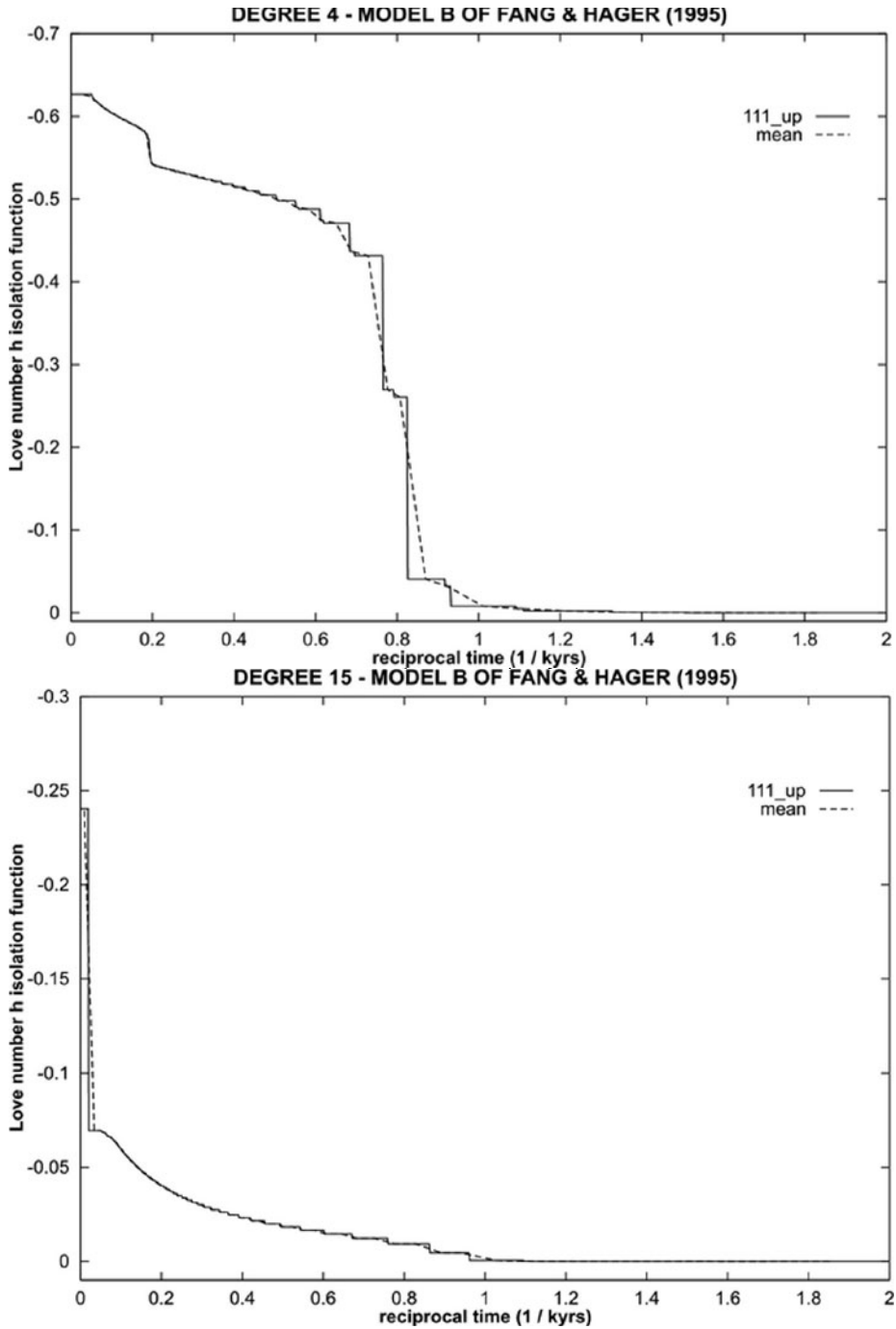


Figure 2.10e. Continued, h Love number for degree 4 with an increased number of layers in the upper mantle. The dashed line in the plots of this panel connects the centers of the time intervals of the solid line.

at about 1.4 1/kyr). The sequence of the modes from right to left for the 5-layer model is the same as in Figure 2.10a. If one looks very carefully, the extra 8-th mode becomes visible also at the extreme left side of the top panel in Figure 2.10b, and is more apparent in the enlargement of the lower panel in Figure 2.10b. If we compare the positions of our modes with the second panel from the top in Fang and Hager's (1995) Figure 9, we conclude that the two buoyancy modes are interchanged in our interpretation with respect to Fang and Hager's. If we consider the various curves in Figure 2.10a, then we see a profound difference between the behavior of the two buoyancy modes on the one hand and the viscoelastic modes on the other. The surface buoyancy mode at about 1.65 1/kyr remains strong when we increase our model from 5 layers to 111 layers. The relaxation time and strength of this buoyancy mode are only slightly changed, which is understandable as the rheological profile is slightly different between the models. The relaxation time and strength of this mode are not only dependent on the density, as Fang and Hager (1995), left column on page 857 suggest, but also on the viscosity. The most direct way to see this is by considering the analytical formulas for the relaxation time and the Love numbers of a homogeneous sphere (see, e.g., equations (33) and (61) in Wu and Peltier 1982). Going from a 5-layer model to a 111-layer model increases the viscosity of the uppermost layer whenever the viscosity profile has been determined by a volume-averaging procedure. Thus one can expect changes in both the relaxation time and the strength of the surface buoyancy mode. Even the buoyancy mode arising from the density contrast at the CMB remains relatively sharp in all the curves in Figure 2.10a, although this sharpness is somewhat covered up by the viscoelastic modes when the number of layers in the model increases. The behavior of the viscoelastic modes is distinctly different from the behavior of the buoyancy modes: they do not remain intact independently of the number of layers, but show a progressively more continuous behavior when the number of layers is increased. This transition from discreteness to continuous behavior is clearly depicted in the enlargements in the Figures 2.10b-c. The lower panels in these figures show the left part of the upper panels. For some parts of these lower panels the discrete character of the 111-layer simulations is hardly visible anymore. That the continuous curves do not completely overlap one another, especially for the l Love number in Figures 2.10c, is attributable to the fact that the volume-averaging procedure does not reach continuum limits when the lithosphere is absent/not elastic. The distinct behavior of the buoyancy modes becomes less clear for the higher harmonic degrees, partly because the strengths of the modes are strongly diminished (compare also the magnitudes on the vertical axes in this respect), and partly because the viscoelastic modes tend to mix with the buoyancy modes in these isolation function plots. Perhaps the most striking evidence that our curves do not differ significantly from Fang and Hager's (1995) comes from Figure 2.10e. The h Love number isolation function

is here plotted for an increased number of layers in the upper mantle: the upper mantle is divided into 109 layers and the lower mantle is homogeneous. With the core, the model again contains 111 layers. The continuous behavior on the left sides of both plots is again clearly visible. Apart from the isolation function as determined with our discrete normal mode method, we have also added a dashed curve which connects the centers of the time-intervals of the solid curve. The philosophy behind this curve is that it would be more representative of the continuous behavior of a Runge-Kutte integration scheme, as such a numerical integration scheme has difficulties in modeling sharp discontinuous jumps. The resemblance between the dashed curves in our Figure 2.10e and the corresponding cases in Fang and Hager's (1995) Figure 9 is striking. The 'saw-tooth' profile in the center of the plots follows almost exactly the same pattern. The only noticeable difference can be seen on the extreme left sides of the plots of degree 15: in our Figure 2.10e the curve jumps sharply, representing the CMB buoyancy mode, while in Fang and Hager's Figure 9 this jump is absent.

The conclusion from these simulations is that the isolation function shows two kinds of distinct behavior in the continuum limit: discontinuous jumps associated with buoyancy modes arising from density contrasts, and continuous behavior associated with the continuous rheological profile. We thus do not find a distinct behavior in relaxation between sets A and B in Figure 7 of Fang and Hager (1995), and we cannot validate their conclusions regarding the failure of the normal mode technique for certain kinds of continuous profiles. Finally, it is of course also interesting to see how the isolation function will behave for a continuous density profile.

In Figure 2.11 the isolation functions are drawn for the convex viscosity profile models. Figure 2.11a contains the plots for the k Love number for degrees 2 and 6, Figure 2.11b the same for the h Love number, and Figure 2.11c the plots for the l Love number. It is clear from these plots that even for a continuous density profile (and at the same time a continuous viscosity profile) parts of the curves converge to a continuous pattern and parts of the curves remain discontinuous. It can also nicely be seen that, especially at the left sides of the plots, the continuous behavior seems to converge to one limiting curve and not to various limiting curves as is the case in the plots of Figure 2.10. This again agrees with the results of the former sections that when a lithosphere is present the Love numbers show saturated behavior when the model contains 30 to 40 layers with a fine-layered shallow part.

Although there remain a few gaps in a complete analytical description of many-layer viscoelastic normal mode models (e.g., analytical proofs that for each boundary between viscoelastic layers exactly three relaxation modes are triggered, and that all poles in the complex plane are poles of first order are still lacking as far as we know), we have not found any indication of fundamental shortcomings in classical normal-mode theory as has been suggested by some

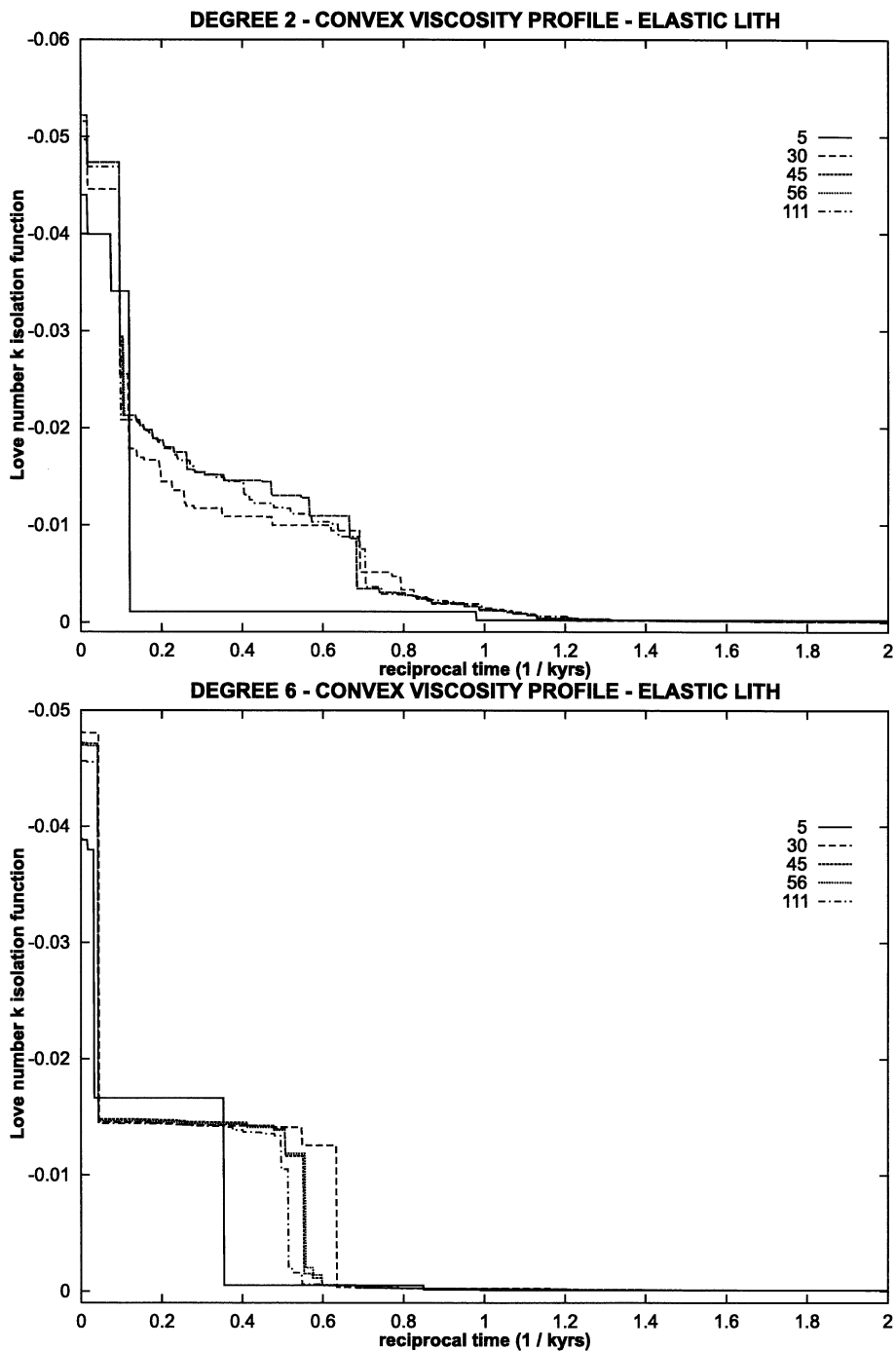


Figure 2.11a. Isolation functions for degree 2 (upper panel) and degree 6 (lower panel) for the convex viscosity profile of Figure 2.3; (a) shows the isolation function for the k Love number, (b) for the h Love number, and (c) for the l Love number. All isolation functions are plotted at time 1.2 kyr. Redrawn from Figure 11 in Vermeersen and Sabadini (1997).

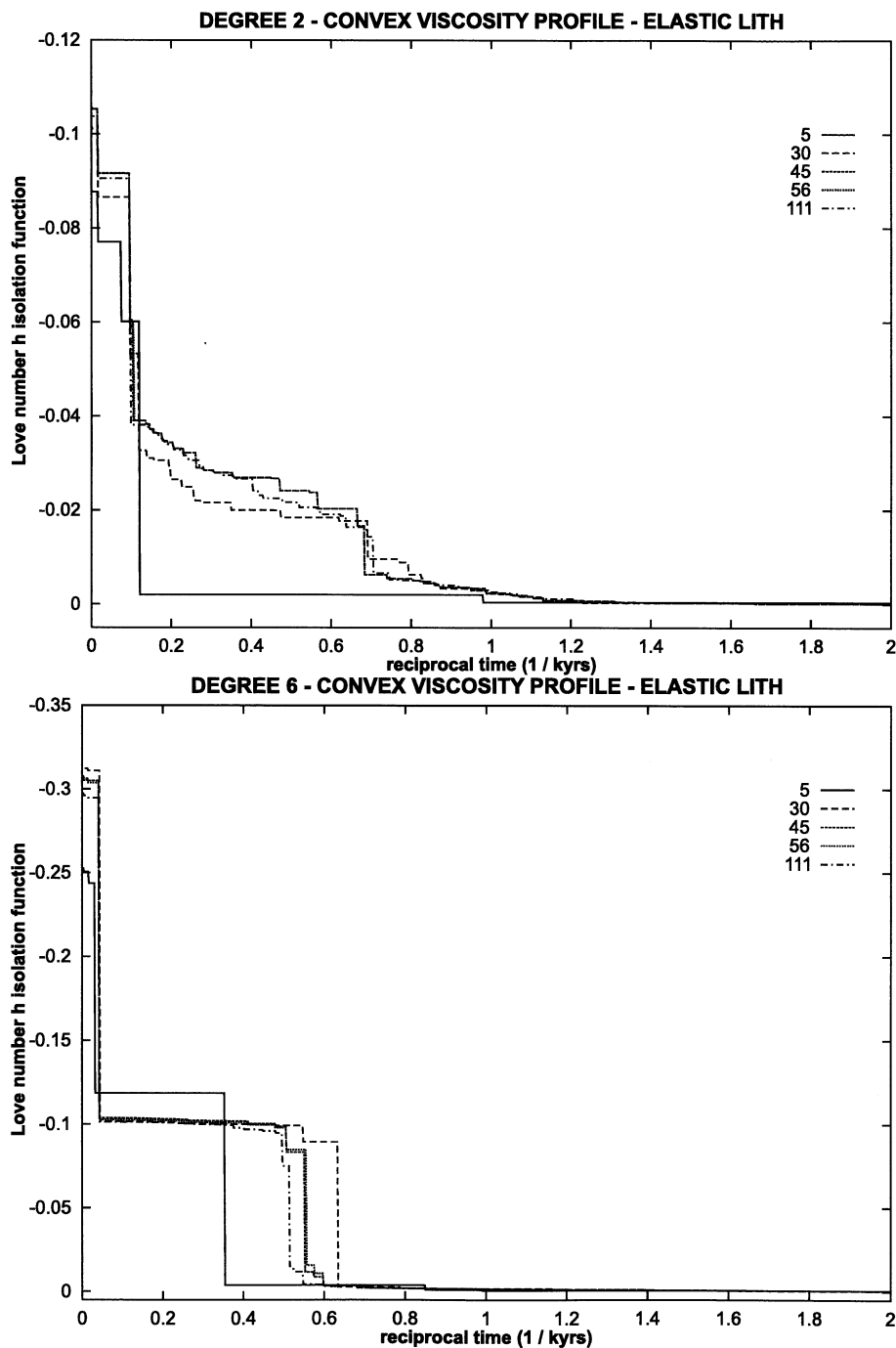


Figure 2.11b. (Continued.)

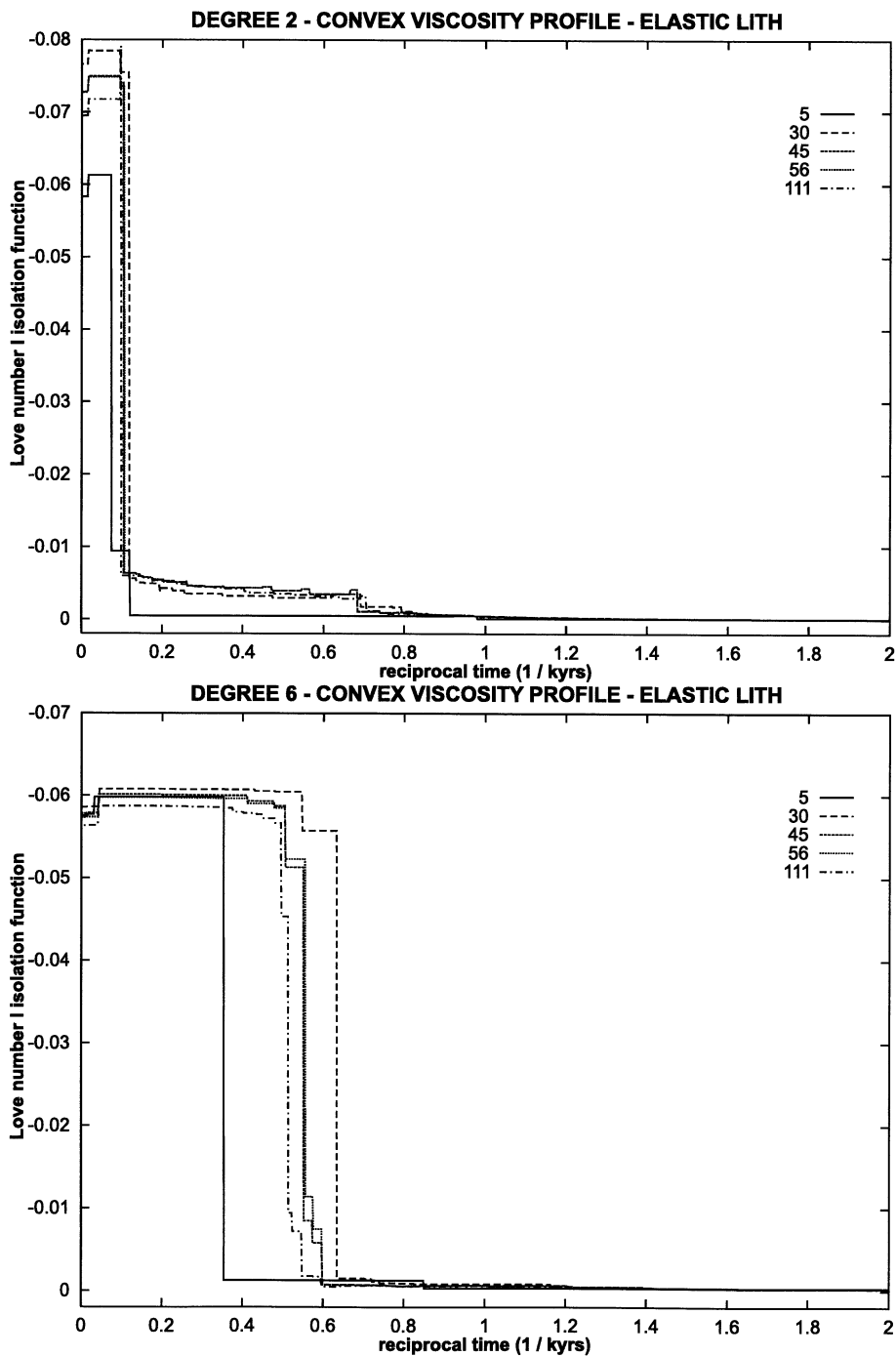


Figure 2.11c. (Continued.)

authors. The expected fluid limits are obtained. On transient timescales our models convert to continuous limits, as testified by the behavior of the isolation functions. Apart from continuous behavior, our isolation functions do show expected discontinuous behavior for the discrete density jumps in the models, like those associated with the surface and the CMB. The question of having reached the continuum limit is thus in our opinion no different from that associated with numerical methods like direct integration of the equations in the time domain or using numerical techniques like finite element or finite difference modeling in general. One should be cautious in this respect that the term 'continuous model' does not refer to the way in which these models are solved. In our opinion, the 'continuous poles' model of Fang and Hager (1995) is only an alternative way of performing an inverse transformation from the Laplace domain to the time domain. The discretized Runge-Kutta scheme they employ to obtain these results for very fine-graded models plays the same role as the discrete normal mode method (Wu, 1978). When 'non-modal contributions' are detected in a model, they are thus a sign of numerical inaccuracies, of not working with complete sets of normal modes, or of not having a fine-graded enough stratification.

Models which have an elastic lithosphere and a convex mantle viscosity profile show saturated (continuum response) behavior for all harmonic degrees up to at least 150 and on all timescales when the number of layers is about 30 to 40. The only exception, most likely due to limits of computational accuracy, occurs in the models with 56 and 111 layers for the higher order degrees in the cases in which a lithosphere is absent. As it is well known that for such high degrees the buoyancy modes shift extremely close to zero and any normal mode technique breaks down near zero when the lithosphere is absent, these accuracy limits are thus rather to be taken as a testimony to the high accuracies which can be obtained with the analytical modeling technique than as a shortcoming. Especially for high degree harmonics, it is important to use a fine-graded shallow layering to obtain continuum limits. The results thus indicate that the best procedure is to first reduce a fine-graded layering to a standard frame by the volume-averaging procedure and then to run the relaxation model. In this way an optimum will be reached between accuracy of the model and reduction of the chance of missing contributing modes in the rootfinding procedure. The model results show further that when one is using a 5-layer model, it is better to use volume-averaged rheological and constitutional parameters than parameters based on the contrasts at the boundaries of fine-graded layer models (e.g., PREM), at least if non-adiabatic density jumps are considered, as discussed in detail when dealing with the polar wander of a viscoelastic Earth.

Chapter 3

ROTATIONAL DYNAMICS OF VISCOELASTIC PLANETS

1. INTRODUCTION TO EARTH ROTATION

The rotation of the Earth is not regular. It changes on virtually every time scale we know in both position of the rotation axis and rotation rate. Even in our daily lives we sometimes experience the consequences of such changes, such as the second that is subtracted or added to clocks at the beginning of a new year. Although this second is not much more than a curiosity for most of us, the rotational changes it implies can influence our lives in a more fundamental sense. There are indications that the emergence of the great ice ages some two million years ago was triggered by a gradual shift of the rotation axis over the Earth's surface, combined with wandering of the continents and associated changes in ocean currents (note that we are talking here about the onset of ice ages - the period of the 100,000 year cycles of ice build-up and decay is determined by astronomical causes). In the 19th century, both the rate of rotation and the position of the rotation axis were shown to be variable. Nowadays we know that these changes occur on all time scales: from shorter than a day to geological ones of hundreds of millions of years. The changes in position of the rotation axis can be divided into two main categories: those in which the position of the axis changes with respect to the distant stars but not with respect to the Earth's surface, and vice versa. For the latter category, it looks to a hypothetical observer in space as though the Earth shifts underneath its rotation axis as a solid unit while the rotation axis itself remains fixed with respect to the stars, while for an observer on Earth it looks as though the rotation axis is wandering over the Earth's surface. Displacements of the axis of rotation with respect to the fixed stars (changes in which the whole planet is moving rigidly as one unit) are mainly due to external forces, notably the gravitational interactions between the Earth and the Sun, the Moon and the other planets

of the solar system. The astronomically well-known precession and nutation are examples of this. The external forces exert a net torque on the equatorial bulge of the Earth, as a consequence of which the rotation axis spins. The most important periods are about 26,000 years (precession) and 18.6 years (nutation)

Displacements of the axis of rotation with respect to a fixed position on the Earth's surface are mainly due to mass displacements in the interior of the Earth and in the hydro- and atmosphere. These mass displacements will generally induce changes in the moments and products of inertia. As the Earth is a deformable body, the rotation axis will readjust itself to the new situation by shifting over the surface. The rotation axis does not change its position with respect to the stars, as during these mass displacements the angular momentum of the Earth is preserved.

Apart from tidal interactions, there are a number of possible mechanisms responsible for the observed rotational variations, like growth and decay of ice sheets, changes in sea level, ocean currents, winds and changes in the pressure distribution of the atmosphere, seasonal changes, earthquakes, tectonic plate movements, changes in convection of mantle and core, and interactions between the core and mantle. Each of these mechanisms operates on specific time scales and this is reflected in the time scales on which the rotation of the Earth changes.

Table 3.1. Variations in the position of the rotation axis with respect to the crust.

type	magnitude	presumed cause
annual wobble	5 m per year	atmospheric pressure
Chandler wobble	10 m per 14 months	not well known
secular drift	1 m per 10 years	glacial rebound, mantle convection

Table 3.1 gives a rough indication of the observed present-day variations in position of the polar axis. The observed changes in the position of the rotation axis with respect to the Earth's surface, generally termed polar wander, consist of two kinds of movements: periodic and linear. The periodic motions consist mainly of two periods. The annual wobble is principally due to seasonally varying zonal winds. The cause of the Chandler wobble is still largely unknown. This periodic movement, which is essentially the free precession of the Earth, was theoretically predicted in the 18th century by Euler (therefore it is also called the Eulerian free precession) but not observed until the end of the 19th century.

The contemporary secular drift, whose direction is shown in Figure 3.1, has been determined by astrometric observation. Post-glacial rebound (Chapter 4) resulting from the decay of these ice sheets is thought to be the main cause of this secular drift, although mantle convection might also make a contribution.

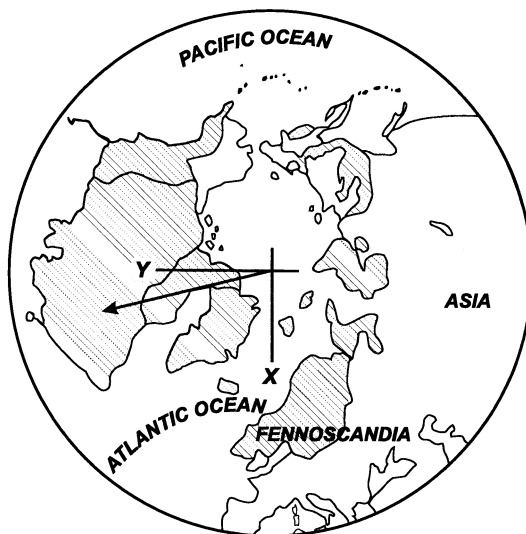


Figure 3.1. Direction of the contemporary secular drift of the axis of rotation. Redrawn from Figure 1 in Vermeersen *et al.* (1997).

The observed changes in the length of day, with the mechanisms that are thought to be responsible, are indicated in Table 3.2.

Table 3.2. Variations in the Length of Day.

time scale	magnitude	presumed cause
daily	milliseconds	atmosphere and tides
annual	milliseconds	seasons
decades	10 ms	core-mantle coupling
> 100 years	2 ms per 100 yrs	tidal interaction, glacial rebound

Short-time fluctuations are very well explained by changes in the pressure distribution over the globe and zonal winds. Figure 3.2 shows the theoretically determined effect of the observed pressure distribution on the Earth's rotation rate, dashed, and the measured length of day variations, solid. The short-time fluctuations correspond almost exactly.

Figure 3.2 also shows that on somewhat longer time scales both lines diverge. Apparently, on these somewhat longer time scales there are other forcing mechanisms at work. Changes on time scales of a few tens of years, the decade fluctuations as depicted in Figure 3.3, are attributable to the exchange of angular momentum between the fluid outer core and the lower mantle, as they appear to be correlated with changes in magnetic field strength. It is these fluctuations

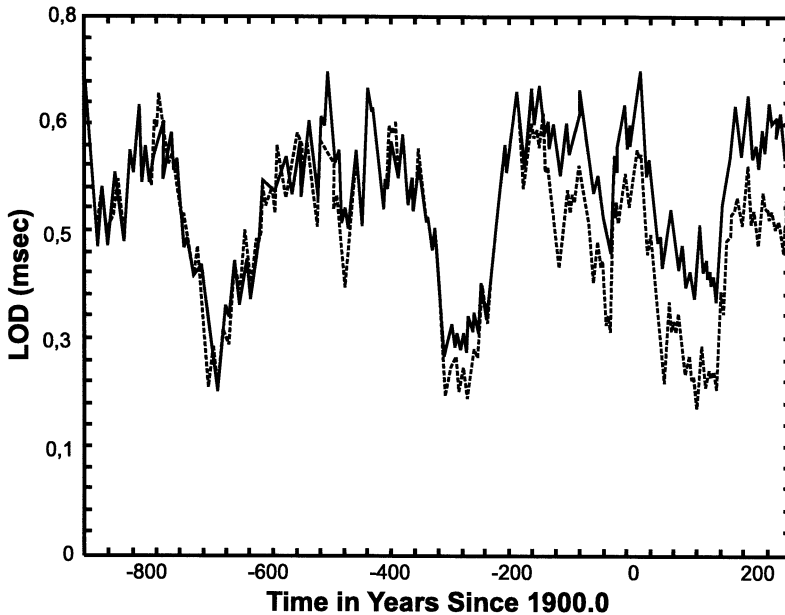


Figure 3.2. Observed and pressure distribution-derived changes in length of day: the solid curve stands for the observation and the dashed one is derived from wind and pressure data. Redrawn from Figure 2 in Wahr (1988).

that are the main cause of why our clocks sometimes need to be adjusted by a second at the end of the year: a 3 ms change in length of day would imply a change in length of year of about one second.

In Figure 3.3 there is also a long-term trend visible (the average rises with time). This is the secular trend, which is mainly due to tidal exchange of angular momentum between the Earth and the Moon and between the Earth and the Sun. The secular decrease of the rotation velocity is due to the fact that the Earth is rotating faster around its axis than the Moon is revolving around the Earth, and because the Earth is not perfectly elastic. As a result, the line connecting the Earth and the Moon does not coincide exactly with the line between the center of the Earth and the place on the surface that shows the maximum tidal height. This gives a net torque, which brakes the Earth's rotation rate and accelerates the Moon in its orbit around the Earth. This is illustrated in the cartoon in Figure 3.4. As the Earth rotates with a progressively slower rate, the centrifugal force diminishes. As a consequence, the ellipticity of the Earth decreases.

This brief overview shows that the variations in the rotation of the Earth form a rich ensemble of geophysical and geodetic phenomena. In this chapter we will first derive the mathematical expressions for polar wander and changes in the length of day for a rigid earth model. Consequently, we will derive the

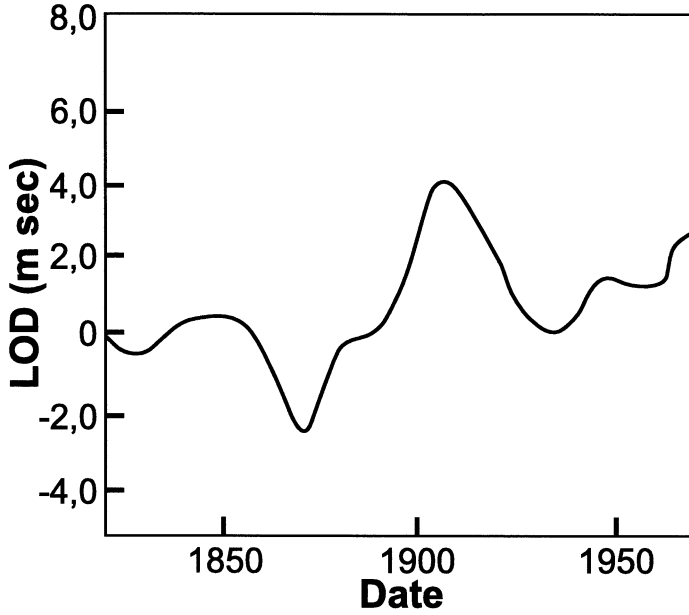


Figure 3.3. Decade fluctuations in the length of day. Redrawn from Figure 1 in Wahr (1988).

mathematical formalism for the viscoelastic adjustment of the equatorial bulge, which plays an important part in any realistic consideration of the rotation of the Earth.

2. ROTATIONAL CHANGES FOR A RIGID EARTH

In this section we assume that the Earth is rigidly rotating and that torques, mass displacements and relative motions can perturb the rotation.

If $\mathbf{L}$ denotes torque, $\mathbf{H}$ angular momentum and $\boldsymbol{\omega}$ angular velocity, then in a reference frame co-rotating with the Earth, Euler's dynamic equation reads

$$\frac{d\mathbf{H}}{dt} + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{L}, \quad (3.1)$$

with

$$\mathbf{H} = \int \rho \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) dV = \int \rho (r^2 \boldsymbol{\omega} - (\mathbf{r} \cdot \boldsymbol{\omega}) \mathbf{r}) dV \quad (3.2)$$

or

$$\mathbf{H} = \mathbf{I}^* \cdot \boldsymbol{\omega} \quad \text{with} \quad I_{ij}^* = \int \rho (r_k r_k \delta_{ij} - r_i r_j) dV \quad (3.3)$$

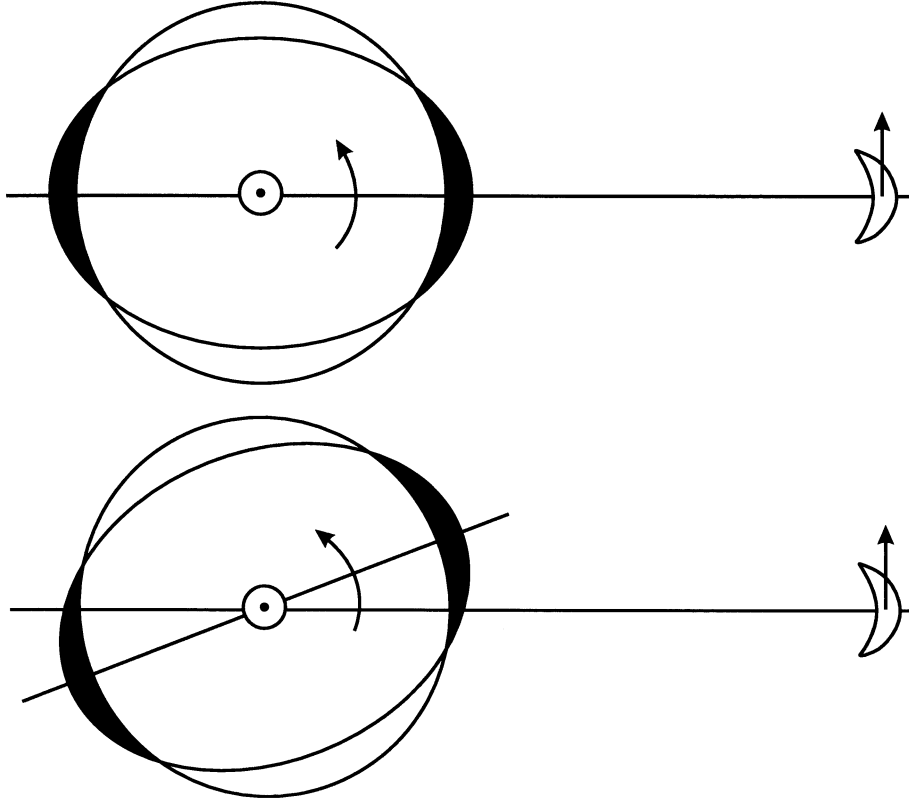


Figure 3.4. Cartoon depicting the mechanism behind tidal deceleration of the Earth's rotation. Redrawn from Figure 11.13 in Munk and MacDonald (1960).

with $\mathbf{r}$ denoting the coordinates of the mass element characterized by the density ρ . The inertia tensor is expressed by $\mathbf{I}^*$; δ_{ij} is the Kronecker delta function. This results in

$$\frac{d}{dt}(\mathbf{I}^* \cdot \boldsymbol{\omega}) + \boldsymbol{\omega} \times (\mathbf{I}^* \cdot \boldsymbol{\omega}) = \mathbf{L}. \quad (3.4)$$

In the general case $\mathbf{I}^*$ is time-dependent because mass displacements and relative motion, $\mathbf{u}$, can occur with respect to the axis $\mathbf{r}$. Therefore $\mathbf{H}(t)$ can be expressed as

$$\mathbf{H}(t) = \mathbf{I}^*(t) \cdot \boldsymbol{\omega} + \mathbf{h}(t), \quad (3.5)$$

in which

$$\mathbf{h}(t) = (h_1, h_2, h_3) = \int \rho \mathbf{r} \times \mathbf{u} dV. \quad (3.6)$$

Substituting equation (3.5) in (3.1) leads to the so-called Liouville equation

$$\frac{d}{dt} (\mathbf{I}^*(t) \cdot \boldsymbol{\omega} + \mathbf{h}(t)) + \boldsymbol{\omega} \times (\mathbf{I}^*(t) \cdot \boldsymbol{\omega} + \mathbf{h}(t)) = \mathbf{L}. \quad (3.7)$$

If Ω denotes the mean Earth rotation frequency, then the components of $\boldsymbol{\omega}$ can be expressed in the dimensionless quantities m_i as

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = \Omega(m_1, m_2, 1 + m_3). \quad (3.8)$$

The quantities m_i are small whenever the deviations from the axis of rotation are small. By assuming that $\mathbf{h}$ and the changes in $\mathbf{I}^*$ are small, the inertia tensor can be written as

$$\mathbf{I}^* = \begin{pmatrix} A + \Delta I_{11}(t) & \Delta I_{12}(t) & \Delta I_{13}(t) \\ \Delta I_{21}(t) & A + \Delta I_{22}(t) & \Delta I_{23}(t) \\ \Delta I_{31}(t) & \Delta I_{32}(t) & C + \Delta I_{33}(t) \end{pmatrix}, \quad (3.9)$$

in which A and C denote the moments of inertia for an equatorial principal axis and the polar principal axis, respectively. We assume that the initial mass distribution of the Earth is symmetric with respect to the rotation axis, so that the moments of inertia for the two principal equatorial axes are both equal to A . It follows from equations (3.8) and (3.9) that, to first order

$$\begin{aligned} \mathbf{I}^* \cdot \boldsymbol{\omega} &= \begin{pmatrix} (A + \Delta I_{11})\omega_1 + \Delta I_{12}\omega_2 + \Delta I_{13}\omega_3 \\ \Delta I_{21}\omega_1 + (A + \Delta I_{22})\omega_2 + \Delta I_{23}\omega_3 \\ \Delta I_{31}\omega_1 + \Delta I_{32}\omega_2 + (C + \Delta I_{33})\omega_3 \end{pmatrix} \\ &\approx \begin{pmatrix} Am_1\Omega + \Delta I_{13}\Omega \\ Am_2\Omega + \Delta I_{23}\Omega \\ C\Omega + Cm_3\Omega + \Delta I_{33}\Omega \end{pmatrix} \end{aligned} \quad (3.10)$$

so

$$\boldsymbol{\omega} \times (\mathbf{I}^* \cdot \boldsymbol{\omega} + \mathbf{h}) \approx \begin{pmatrix} Cm_2\Omega^2 - Am_2\Omega^2 - \Delta I_{23}\Omega^2 - h_2\Omega \\ Am_1\Omega^2 + \Delta I_{13}\Omega^2 + h_1\Omega - Cm_1\Omega^2 \\ 0 \end{pmatrix} \quad (3.11)$$

resulting, by substituting equations (3.10) and (3.11) in (3.7), with $\mathbf{L} = (L_1, L_2, L_3)$, in

$$\begin{aligned} A\Omega\dot{m}_1 + (C - A)\Omega^2m_2 &= -\Omega\Delta\dot{I}_{13} + \Omega^2\Delta I_{23} \\ &\quad + \Omega h_2 - \dot{h}_1 + L_1 \end{aligned} \quad (3.12)$$

$$\begin{aligned} A\Omega\dot{m}_2 - (C - A)\Omega^2m_1 &= -\Omega\Delta\dot{I}_{23} - \Omega^2\Delta I_{13} \\ &\quad - \Omega h_1 - \dot{h}_2 + L_2 \end{aligned} \quad (3.13)$$

$$C\Omega^2\dot{m}_3 = -\Omega^2\Delta\dot{I}_{33} - \Omega\dot{h}_3 + \Omega L_3, \quad (3.14)$$

where the dot denotes the time derivative. With the Eulerian free precession frequency defined as

$$\sigma_r = \frac{C - A}{A} \Omega, \quad (3.15)$$

this can be written as

$$\frac{\dot{m}_1}{\sigma_r} + m_2 = \phi_2 \quad (3.16)$$

$$\frac{\dot{m}_2}{\sigma_r} - m_1 = -\phi_1 \quad (3.17)$$

$$\dot{m}_3 = \dot{\phi}_3, \quad (3.18)$$

in which

$$\phi_1 = \frac{1}{(C - A)\Omega^2} (\Omega^2 \Delta I_{13} + \Omega \Delta \dot{I}_{23} + \Omega h_1 + \dot{h}_2 - L_2) \quad (3.19)$$

$$\phi_2 = \frac{1}{(C - A)\Omega^2} (\Omega^2 \Delta I_{23} - \Omega \Delta \dot{I}_{13} + \Omega h_2 - \dot{h}_1 + L_1) \quad (3.20)$$

$$\phi_3 = \frac{1}{C\Omega^2} (-\Omega^2 \Delta I_{33} - \Omega h_3 + \Omega \int_0^t L_3 dt') \quad (3.21)$$

are the dimensionless excitation functions.

If we only consider mass displacements, neglecting influences of relative motions ($\mathbf{h}, \dot{\mathbf{h}}$) and time variations of the inertia ($\Delta \dot{I}_{ij} = 0$), and in the absence of external torques ($\mathbf{L} = 0$), then these excitation functions reduce for polar wander to the following excitation function written in complex notation

$$\Phi_L = \phi_1 + i\phi_2 = \frac{\Delta I_{13}}{C - A} + i \frac{\Delta I_{23}}{C - A}. \quad (3.22)$$

As in a deformable Earth there are no stable reference frames in the strict sense, it is necessary to define such a reference system in a practical way. For short-term polar wander it is convenient to take the geographical frame as reference frame. It can be defined as the mean position of a number of fixed points in stable continental areas. For long-term true polar wander the choice of a reference frame becomes more complicated, as the whole mantle can change its configuration. Usually, the hot-spot reference frame is taken as the frame in which the mean mantle material is stable. The relative velocities between the hot spots are generally found to be a factor of one tenth smaller than the relative plate velocities. m_1 and m_2 now give the resultant polar shift in radians: m_1 in the x -direction, which is chosen to be in the equatorial plane from the center of the Earth towards the Greenwich meridian, and m_2 in the y -direction, which

is chosen to be in the equatorial plane from the center of the Earth towards 90 degrees East longitude. m_3 gives the change in the length of day in radians per sidereal day.

With this and the polar shift in complex notation $\mathbf{m} = m_1 + im_2$, the linearized Liouville equation for polar wander can be written from equations (3.16), (3.17) and (3.22) in complex notation, with i the imaginary unit, as

$$i \frac{\dot{\mathbf{m}}}{\sigma_r} + \mathbf{m} = \Phi_L. \quad (3.23)$$

For loadings that change with much smaller frequencies than σ_r , the linearization of the Liouville equation leads ultimately to

$$m_1 = \frac{\Delta I_{13}}{C - A} \text{ rad } x - \text{component of polar shift} \quad (3.24)$$

$$m_2 = \frac{\Delta I_{23}}{C - A} \text{ rad } y - \text{component of polar shift} \quad (3.25)$$

$$m_3 = \frac{-\Delta I_{33}}{C} \text{ rad } \text{change in length of day.} \quad (3.26)$$

Note that the equatorial flattening is of great importance: if C were equal to A , then the Eulerian free precession frequency, equation (3.15), would be zero, the excitation functions, equations (3.19) and (3.20), would be infinite and, according to equations (3.24) and (3.25), the polar shift infinitely large.

Exercise. For the Earth, $C = 8.0394 \times 10^{37} \text{ kg m}^2$ and $A = 8.0131 \times 10^{37} \text{ kg m}^2$. Determine the period of the Eulerian free precession frequency.

Another important issue is that equations (3.16) and (3.17), or (3.24) and (3.25), do not take a shift in the equatorial bulge into account. To put it differently: (3.16) and (3.17), or (3.24) and (3.25), give the polar wander for a rigid planet. If the rotation axis were to coincide with the axis perpendicular to the plane of the equatorial flattening before a mass change occurs, then after the mass change equations (3.16) and (3.17), or (3.24) and (3.25), would give the new position of the rotation axis that coincides with the axis of maximum moment of inertia. This new position would not be perpendicular to the plane of the equatorial flattening whenever the perturbing mass is not on the poles or the equator. A wobble ensues, which should have the frequency given by equation (3.15). At the end of the 19th century astronomers looked for this frequency in their observations but did not find it. However, they found a strong wobble, the so-called Chandler wobble, that had a period four months greater than the Eulerian free precession period. It was soon realized that the Chandler wobble is nothing else than the Eulerian free precession and that the four-month period extension is due to the elastic properties of the Earth. The deformation of the Earth is also responsible for the decay of the wobble amplitudes on time scales

of a few decades, implying that the wobbles must be maintained by geophysical forcings. And, as we have already seen in the introduction, the tidal deceleration of the Earth's rotation also requires that (visco)elastic adjustment of the equatorial bulge be taken into account. Finally, for long-term polar wander it is obvious that the shift of the equatorial bulge has to be taken into account. There are indications that polar wander over geological times may amount up to several tens of degrees, while today the equatorial bulge is almost perpendicular to the Earth's rotation axis.

3. ADJUSTMENT OF THE EQUATORIAL BULGE

Consider the angle α between the instantaneous rotation axis $\boldsymbol{\omega}$ and the line $\mathbf{r}$ from the center of mass of the Earth and the observation point P; we thus have

$$\boldsymbol{\omega} \cdot \mathbf{r} = |\boldsymbol{\omega}| |\mathbf{r}| \cos \alpha, \quad (3.27)$$

with $\boldsymbol{\omega} \cdot \mathbf{r} = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3$.

The distance l from P perpendicular to the axis of rotation now follows from

$$l^2 = r^2 - r^2 \cos^2 \alpha = r^2 - \frac{(x_1 \omega_1 + x_2 \omega_2 + x_3 \omega_3)^2}{\omega^2}, \quad (3.28)$$

with $r = |\mathbf{r}|$ and $\omega = |\boldsymbol{\omega}|$.

So with equation (3.28) the centrifugal potential $U_c = \frac{1}{2} \omega^2 l^2$ is given by

$$U_c = \frac{1}{2} \omega^2 l^2 = \frac{1}{2} [\omega^2 r^2 - (\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3)^2]. \quad (3.29)$$

Exercise. Show that equation (3.29) can be written as

$$U_c = \frac{1}{3} \omega^2 r^2 + \Delta U_c, \quad (3.30)$$

with $\frac{1}{3} \omega^2 r^2$ the purely radial part of U_c , and ΔU_c the part which describes the equatorial flattening

$$\begin{aligned} \Delta U_c = & \frac{1}{6} [\omega_1^2 (x_2^2 + x_3^2 - 2x_1^2) + \omega_2^2 (x_1^2 + x_3^2 - 2x_2^2) \\ & + \omega_3^2 (x_1^2 + x_2^2 - 2x_3^2) - 6\omega_1 \omega_2 x_1 x_2 \\ & - 6\omega_1 \omega_3 x_1 x_3 - 6\omega_2 \omega_3 x_2 x_3]. \end{aligned} \quad (3.31)$$

Exercise. Show that ΔU_c is a spherical harmonic of degree 2.

The deformation of the Earth caused by this degree 2 centrifugal potential ΔU_c gives rise to an exterior gravitational potential

$$V = k_2(a^5/r^5)\Delta U_c, \quad (3.32)$$

where k_2 denotes the tidal Love number, equation (1.137) for $l = 2$, obtained from the boundary conditions given in Chapter 1 by equation (1.127) for tidal loading; in rotational problems, only the degree two harmonic component of the deformation plays a role, as indicated by the subscript 2 in the tidal Love number.

Near the boundary of a body departing slightly from spherical symmetry, the degree 2 contribution of the gravitational potential can be arranged in terms of the perturbation in the moments of inertia ΔI_{ij} , according to MacCullagh's formula (Jeffreys, 1976), which gives

$$\begin{aligned} V = \frac{G}{2r^5} [& (\Delta I_{11}(x_2^2 + x_3^2 - 2x_1^2) + \Delta I_{22}(x_1^2 + x_3^2 - 2x_2^2) \\ & + \Delta I_{33}(x_1^2 + x_2^2 - 2x_3^2) - 6\Delta I_{12}x_1x_2 - 6\Delta I_{13}x_1x_3 \\ & - 6\Delta I_{23}x_2x_3)]. \end{aligned} \quad (3.33)$$

From the equations above, the changes in the products of inertia ΔI_{13} and ΔI_{12} can be expressed in a linear approximation as

$$\Delta I_{13} = \frac{k_2 a^5}{3G} \omega_1 \omega_3 \approx \frac{k_2 a^5 \Omega^2}{3G} m_1 (1 + m_3) \quad (3.34)$$

$$\Delta I_{23} = \frac{k_2 a^5}{3G} \omega_2 \omega_3 \approx \frac{k_2 a^5 \Omega^2}{3G} m_2 (1 + m_3) \quad (3.35)$$

and, as $m_3 \ll 1$, the forcing function Φ_R for rotational deformation can be written in complex notation as in equation (3.22) as

$$\Phi_R = \frac{\Delta I_{13}}{C - A} + i \frac{\Delta I_{23}}{C - A} \approx \frac{k_2}{k_f} \mathbf{m}, \quad (3.36)$$

with $\mathbf{m} = m_1 + im_2$, and

$$k_f = \frac{3G(C - A)}{a^5 \Omega^2} \quad (3.37)$$

the so-called secular Love number.

The linearized Liouville equation for polar wander (3.23), which includes both loading and tidal forcings, can be written as

$$i \frac{\dot{\mathbf{m}}}{\sigma_r} + \mathbf{m} = \Phi, \quad (3.38)$$

with the forcing function Φ consisting of two parts: $\Phi = \Phi_L + \Phi_R$, with Φ_L the part describing the direct geodynamic forcing and Φ_R the induced rotational deformation.

The linearized Liouville equation can thus be expressed as

$$i \frac{\dot{\mathbf{m}}}{\sigma_r} + \left(1 - \frac{k_2}{k_f}\right) \mathbf{m} = \Phi_L \quad (3.39)$$

or

$$i \frac{\dot{\mathbf{m}}}{\sigma_0} + \mathbf{m} = \Psi_L, \quad (3.40)$$

with

$$\sigma_0 = \left(1 - \frac{k_2}{k_f}\right) \sigma_r \quad (3.41)$$

and

$$\Psi_L = \frac{k_f}{k_f - k_2} \Phi_L. \quad (3.42)$$

The term σ_0 is the frequency of the Chandler wobble. Its four-month decrease with respect to the frequency of the Eulerian free precession frequency σ_r is apparently due to the quotient k_2/k_f .

Exercise. Show, assuming that k_2 is not time-dependent, that the solution to (3.40) is

$$\mathbf{m}(t) = -i\sigma_0 e^{i\sigma_0 t} \int_{-\infty}^t \Psi_L(\tau) e^{-i\sigma_0 \tau} d\tau, \quad (3.43)$$

and determine $\mathbf{m}(t)$ for the following two cases:

- (1) $\Psi_L(t) = \Psi_0 H(t)$, with Ψ_0 time-independent and H the Heaviside function;
- (2) $\Psi_L(t) = \Psi_0 \delta(t)$, with Ψ_0 being time-independent and δ the delta function.

Draw your solutions in the (m_1, m_2) -plane for both cases (1) and (2). Assume that at time $t = 0$ the rotation pole is at the origin.

4. DEVELOPMENTS OF LINEARIZED ROTATION THEORIES

The theory of relating changes in mass distribution to changes in rotation for a viscoelastic Earth has been developed over many years. Burgers (1955), acknowledging Gold (1955) as source of inspiration, built such a model in which

the rheology consisted of a variation of the Kelvin-Voigt model (the Newtonian element was replaced by a Maxwell element). References to further early attempts can be found in Munk and MacDonald (1960). Multi-layer viscoelastic relaxation models for Earth rotation driven by ice cycles were built by Sabadini *et al.* (1982a) and Wu and Peltier (1984). The simulations we present are based upon the theory of Sabadini *et al.* (1982a), but the theory of Wu and Peltier (1984) is equivalent to Sabadini *et al.* (1982a) after the Chandler wobble is filtered out in the theory of the latter (Sabadini *et al.*, 1984b; Vermeersen and Sabadini, 1996; Mitrovica and Milne, 1998).

Laplace transformation of (3.39) and substitution of equation (1.137) appropriate for tidal loading lead to

$$\left(i \frac{s}{\sigma_r} + 1 - \frac{1}{k_f} \left(k_e + \sum_{i=1}^M \frac{k_i}{s - s_i} \right) \right) \mathbf{m}(s) = \Phi_L(s), \quad (3.44)$$

with $k_f = k_2(s = 0)$ being the tidal fluid Love number. In agreement with the convention already adopted, $\mathbf{m}(s)$ and $\Phi_L(s)$ are Laplace-transformed quantities. With the tidal fluid Love number

$$k_f = k_e + \sum_{i=1}^M \frac{k_i}{-s_i} \quad (3.45)$$

and

$$\frac{k_i}{s_i} + \frac{k_i}{s - s_i} = \frac{k_i s}{s_i(s - s_i)}, \quad (3.46)$$

equation (3.44) becomes

$$s \left(1 + i \frac{\sigma_r}{k_f} \sum_{i=1}^M \frac{k_i}{s_i(s - s_i)} \right) \mathbf{m}(s) = -i \sigma_r \Phi_L(s). \quad (3.47)$$

So

$$s \left(1 + \sum_{i=1}^M \frac{x_i}{s - s_i} \right) \mathbf{m}(s) = -i \sigma_r \Phi_L(s) \quad (3.48)$$

with

$$x_i = i \frac{\sigma_r k_i}{k_f s_i}. \quad (3.49)$$

Now

$$1 + \sum_{i=1}^M \frac{x_i}{s - s_i} = \frac{\prod_{j=1}^M (s - s_j)}{\prod_{j=1}^M (s - s_j)} + \sum_{i=1}^M x_i \frac{\prod_{j \neq i}^M (s - s_j)}{\prod_{j=1}^M (s - s_j)}, \quad (3.50)$$

where $\prod_{j \neq i}^M$ means $\prod_{j=1}^M$ without the term $j = i$.

The right-hand side of (3.50)

$$\frac{\prod_{j=1}^M (s - s_j) + \sum_{i=1}^M x_i \prod_{j \neq i}^M (s - s_j)}{\prod_{j=1}^M (s - s_j)} \quad (3.51)$$

can be transformed into

$$\frac{\sum_{i=0}^M \alpha_i s^i}{\prod_{j=1}^M (s - s_j)} \quad (3.52)$$

(whereby it immediately follows that $\alpha_M = 1$), and consequently the part in brackets of equation (3.48) can be put as

$$1 + \sum_{i=1}^M \frac{x_i}{s - s_i} = \frac{\prod_{j=1}^M (s - a_j)}{\prod_{j=1}^M (s - s_j)}, \quad (3.53)$$

with a_i being the M complex roots of the equation

$$\sum_{i=0}^M \alpha_i s^i = 0. \quad (3.54)$$

Inserting equation (3.53) in (3.48) gives

$$s \frac{\prod_{j=1}^M (s - a_j)}{\prod_{j=1}^M (s - s_j)} \mathbf{m}(s) = -i\sigma_r \Phi_L(s) \quad (3.55)$$

and so

$$\begin{aligned} \mathbf{m}(s) &= -i\sigma_r \frac{\prod_{j=1}^M (s - s_j)}{s \prod_{j=1}^M (s - a_j)} \Phi_L(s) \\ &= -i\sigma_r \left(\frac{A_0}{s} + \sum_{j=1}^M \frac{A_j}{s - a_j} \right) \Phi_L(s). \end{aligned} \quad (3.56)$$

In this expression, the terms a_j are the inverse relaxation times from the tidal problem for the M modes, having the strength given by the residues A_j . The residue A_0 gives the strength of the secular term. $\mathbf{m}(s)$ is defined in such a way that the real-valued component gives the polar wander in the direction of the

Greenwich meridian, and the imaginary-valued component the polar wander in the direction 90 degrees to the east.

Equation (3.56) forms the basis of realistic models of Earth rotation that take solid-Earth deformation and its consequential shifts in the equatorial bulge self-consistently into account (with one restriction: the Liouville expression has been linearized and therefore polar wander needs to be restricted to about 10 degrees over the Earth's surface, or about 1000 km, at most).

The loading term entering equation (3.56) can be made explicit to give

$$\mathbf{m}(s) = -i\sigma_r \frac{\Delta I_{13}^R + i\Delta I_{23}^R}{C - A} \left(\frac{A_0}{s} + \sum_{j=1}^M \frac{A_j}{s - a_j} \right) \left(1 + k_e^L + \sum_{j=1}^M \frac{k_j^L}{s - s_j} \right) \tilde{f}(s), \quad (3.57)$$

where C and A are the principal moments of inertia and $\tilde{f}(s)$ denotes the Laplace-transformed loading history. The superscript R in the products of inertia indicates that these terms refer to the perturbations of the inertia of the rigid Earth, since the elastic and viscoelastic contribution is included in the terms within the brackets. k_e^L is the elastic load Love number (see equation (1.139)), and the terms k_j^L are the load Love numbers of the M modes j (see equation (1.138)), accompanied by their (negative) inverse relaxation times s_j of Chapter 1; the superscript L is used to differentiate the loading Love number from the tidal one. In this book, perturbations in rotation are due to surface or internal mass anomalies, not to earthquakes. As already observed to the tidal Love number, only the degree two harmonic component of the loading Love number enters the rotation equations.

The rigid-Earth perturbations in the product-of-inertia components ΔI_{13}^R and ΔI_{23}^R for ice loading are given in equation (28) of Wu and Peltier (1984) for the case of a circular disk load with mass M_I and radius α at glacial maximum, having its center at co-latitude θ and eastern longitude ϕ , and for which the melted ice is redistributed over the oceans with a realistic ocean function

$$\Delta I_{13}^R = -M_I a^2 \left(\frac{1}{6} \cos \alpha (1 + \cos \alpha) P_{21}(\cos \theta) \cos \phi - \frac{1}{5} \frac{a_{21}}{a_{00}} \right) \quad (3.58)$$

$$\Delta I_{23}^R = -M_I a^2 \left(\frac{1}{6} \cos \alpha (1 + \cos \alpha) P_{21}(\cos \theta) \sin \phi - \frac{1}{5} \frac{b_{21}}{a_{00}} \right) \quad (3.59)$$

For ΔI_{33}^R we have

$$\Delta I_{33}^R = M_I a^2 \left(\frac{2}{15} \frac{a_{20}}{a_{00}} - \frac{\cos \alpha (1 + \cos \alpha)}{3} P_{20}(\cos \theta) \right). \quad (3.60)$$

The quotients a_{20}/a_{00} , a_{21}/a_{00} and b_{21}/a_{00} are determined from the expansion of the ocean function in spherical harmonics (e.g., Lambeck, 1980). P_{20} , P_{21} are second degree Legendre functions. In our models the ice water is redistributed eustatically over the oceans. Mitrovica and Peltier (1993) report that for the time derivative of the harmonic degree 2 geopotential component, differences of 10 - 15 per cent can be expected with those models which include the redistribution of the meltwater as gravitationally self-consistent.

Equation (3.57) can be further simplified to (e.g., Vermeersen *et al.*, 1994):

$$\mathbf{m}(s) = -i\sigma_r \frac{\Delta I_{13}^R + i\Delta I_{23}^R}{C - A} \left(\frac{A_0^*}{s} + \sum_{i=1}^M \frac{\beta_i}{s - s_i} + \sum_{i=1}^M \frac{\gamma_i}{s - a_i} \right) \tilde{f}(s), \quad (3.61)$$

in which

$$\beta_i = A_0 \frac{k_i^L}{s_i} + \sum_{j=1}^M \frac{A_j k_i^L}{s_i - a_j}, \quad (3.62)$$

$$\gamma_i = A_i(1 + k_e^L) - \sum_{j=1}^M \frac{A_i k_j^L}{s_j - a_i} \quad (3.63)$$

and

$$A_0^* = A_0(1 + k_f^L), \quad (3.64)$$

in which the fluid limit ($s = 0$) of the load Love number is given by

$$k_f^L = k_e^L - \sum_{i=1}^M \frac{k_i^L}{s_i}. \quad (3.65)$$

Exercise. Show that the coefficients β_i are equal to zero.

Exercise. Taking into account that for an incompressible earth model $I_{11} = I_{22} = -I_{33}/2$ and that $J_2 = (C - A)/M_E a^2$ denotes the dynamic form factor, show that the time-derivative $\dot{J}_2$, the normalized perturbation in the geopotential of degree $l = 2$, $-\phi_2$ of equation (1.131), for an incompressible uniform sphere becomes in the Laplace domain

$$\tilde{J}_2(s) = \frac{3}{2M_E a^2} \left(\frac{\bar{\mu}s}{(1 + \bar{\mu})(s + \gamma)} \right) \Delta I_{33}^R \times s \tilde{f}(s), \quad (3.66)$$

where ΔI_{33}^R denotes the rigid perturbation in the moment of inertia defined above, with the adimensional rigidity $\bar{\mu} = \frac{19\mu}{2\rho A a^2}$ and $\gamma = \frac{\mu/\nu}{1 + \bar{\mu}}$, with $A = \frac{4}{3}\pi G\rho$.

Hint: the term in brackets in $\tilde{J}_2$ denotes the quantity $(1 + k_2^L)$ for an incompressible uniform Earth, with k_2^L the loading Love number of degree 2, with 1 denoting the direct effect of the load and k_2^L the perturbation in the gravitational potential due to Earth's deformation. It can be derived from the part of the fundamental solution (1.74) which is regular in $r = 0$, since we have assumed an earth model consisting of a uniform sphere, once the boundary conditions (1.126) are used to match (1.125) at the Earth's surface.

Exercise. Assuming for the uniform earth model $a = 6.371 \times 10^6$ m, $\rho = 5517$ kg/m³, $g = 9.8$ m/s² and $M_E = 5.976 \times 10^{24}$ kg, evaluate the mantle viscosity ν so that it fits the modeled $\tilde{J}_2$ with the observed value -2.9×10^{-11} yr⁻¹ obtained by Devoti *et al.* (2001), with the mass of the ice load being $M_I = 2 \times 10^{20}$ kg, assuming for simplicity $\theta = 0^\circ$. (Answer: $\nu = 1.8 \times 10^{21}$ Pa s).

4.1 COMPARISON BETWEEN DIFFERENT ROTATION THEORIES

As elucidated by equation (3.56), in the case of polar wander there are two relaxation mechanisms at work: load relaxation, as a consequence of a redistribution of ice and water over the Earth's surface, and tidal-effective (or centrifugal) relaxation, as a consequence of the centrifugal force acting on the rotating Earth. This centrifugal force causes the equatorial bulge of the Earth to be displaced over the Earth's surface in a manner not unlike a wave traveling over the ocean's surface. This 'polar wander' movement goes on until the Earth's rotation axis coincides with the axis of maximum moment of inertia of the mass distribution.

The theory that is used to study changes in the second degree harmonic of the geoid $\tilde{J}_2$ and polar wander can be found in a number of publications over the past years (e.g., Nakiboglu and Lambeck, 1980; Sabadini and Peltier, 1981; Sabadini *et al.*, 1982a, 1984b, 1988, 1993; Wu and Peltier, 1984; Peltier, 1985; Spada *et al.*, 1992a; Ricard *et al.*, 1992, 1993; Mitrovica and Peltier, 1993; Vermeersen *et al.*, 1994, 1996b, 1997; Peltier and Jiang, 1996). The models in all these references employ a viscoelastic Maxwell rheology for a spherical earth model (that is, normal mode theory is first applied to a non-rotating spherical earth model, after which the required rotating ellipsoidal earth model is obtained by applying the centrifugal potential). Differences in the models in the above references exist in, for example, the number of layers that the earth model has, the way in which the differential equations are solved (analytically or numerically), whether the Lamé parameter λ is taken as finite or infinite (compressible or incompressible) and whether the models allow only for surface loads or also for internal mantle loads.

The theories developed by Sabadini *et al.* (1982a) and by Wu and Peltier (1984) have been shown to be equivalent in Sabadini *et al.* (1984b) and Vermeersen and Sabadini (1996). Specifically, Sabadini *et al.* (1984b) have shown that the secular polar wander terms are the same, while Vermeersen and Sabadini (1996) have demonstrated that when the Chandler wobble is filtered from the model of Sabadini *et al.* (1982a), then to a high approximation the same polar wander curves are obtained compared to those found with the model of Wu and Peltier (1984) for all timescales.

In Spada *et al.* (1992b), an analytical theory is developed for polar wander and $\dot{J}_2$ models for Earth stratifications with 5 layers at most. Vermeersen *et al.* (1996a) and Vermeersen and Sabadini (1996) have developed the analytical theory for the relaxation of an earth model consisting of a general number of layers. In this section we concentrate on those aspects that are specifically associated with studying rotational changes by means of analytical models in which the Earth is radially stratified with a general number of layers.

4.2 OMISSION OF THE M_0 ROTATION MODE

A number of authors have reported during the past decade that studies on glacially induced displacement of the axis of rotation with respect to the Earth's surface named True Polar Wander, now on TPW, allow in general for multiple solutions for the lower mantle viscosity if the TPW is known and all the other rheological, elastic and constitutional parameters are fixed (Figure 20 in Yuen *et al.*, 1986; Figure 5 in Spada *et al.*, 1992b; Figure 1 in Milne and Mitrovica, 1996). These multiple-branch solutions are found in other geophysical signatures related to glacially induced solid-Earth deformation like post-glacial rebound, free-air gravity anomalies, and changes in the non-tidal acceleration of the Earth. We will come back to these points in detail in the following Chapter 4.

In a recent study (Peltier and Jiang, 1996, Figure 7), such multi-branch solutions for the lower mantle viscosity are conspicuously absent, only allowing for solutions in which the lower mantle is slightly more viscous than the upper mantle. Although the theories described by Sabadini *et al.* (1982a) and Wu and Peltier (1984) appear to have a number of differences, Sabadini *et al.* (1984b) have shown that the formulations are equivalent to some extent. The proof of the equivalence of equations (16) and (17) in Sabadini *et al.* (1984b) for the secular rotation term is an important result in this respect. Another result mentioned in Sabadini *et al.* (1984b) is that each of the load relaxation modes has a corresponding rotational relaxation mode. This correspondence remains an important issue since, in Wu and Peltier's (1984) theoretical development, one of the corresponding modes, the M_0 rotational relaxation mode, is lacking (see also Table 1 in Sabadini *et al.*, 1984b). Peltier and Jiang (1996) extend the analysis of Wu and Peltier (1984) to consider a basic state in which the two

principal equatorial moments of inertia are distinct. The extension is, however, unimportant in predictions of TPW (Peltier and Jiang, 1996). More importantly, the extended theory retains the Chandler wobble filtering procedure of Wu and Peltier (1984) and this raises the possibility that the different behavior of the TPW curves in Peltier and Jiang (1996) is due to the neglect of the $M0$ rotational relaxation mode. We show explicitly that the absence of the $M0$ rotational relaxation mode cannot be the cause of this difference in TPW-rate behavior. As support for this result, it should be noted that the TPW predictions of Milne and Mitrovica (1996), Figure 1, which agree with earlier analyses (e.g., Spada *et al.*, 1992b), are based on theory which is equivalent to Wu and Peltier's (1984) approach. In this section it is first shown that Wu and Peltier's (1984) approach to deleting the Chandler wobble acts to remove the $M0$ rotational mode. A new analytical approximation formula for the $M0$ rotational relaxation mode, which incorporates the Chandler wobble frequency for a stratified Earth as the imaginary part, is then derived. It will be shown, using numerical tests, that this approximation formula is extremely accurate. It is shown that the model approach used in Sabadini *et al.* (1982a) and Wu and Peltier (1984) lead to the same secular TPW results.

Equation (64) in Wu and Peltier (1984) becomes in our notation

$$\mathbf{m}(s) = -\frac{i\sigma_r \Phi_L(s)}{s \left(1 + i \sum_{j=1}^M \frac{x'_j}{s - s_j} \right)}. \quad (3.67)$$

The terms $x'_i = (\sigma_r k_i)/(k_f s_i)$ do not contain the imaginary unit i as the previously defined x_i in equation (3.49) do and have the same dimension as s_j (parameters k_j are the same as the parameters t_j used in Wu and Peltier, 1984).

For inverse Laplace transformation of equation (3.67) the complex-valued roots of the denominator on the right hand side have to be found. At this stage, Wu and Peltier (1984) make the point that the unity term in the denominator of (3.67) may be neglected. This is only correct, however, if the imaginary parts of the roots have the same magnitude or are much smaller than the magnitude of the real parts. This is indeed the case for $M - 1$ roots, but it is not true for one root which has a much larger imaginary part than real part. This root turns out to be the rotational root that gives the relaxation of the fundamental mantle mode as real part and the Chandler wobble as imaginary part.

The omission of this rotational root in Wu and Peltier (1984) will become more apparent when we rewrite the term inside the brackets of equation (3.67) as

$$1 + i \sum_{j=1}^M \frac{x'_j}{s - s_j} = \frac{\prod_{j=1}^M (s - s_j) + i \sum_{k=1}^M x'_k \prod_{j \neq k}^M (s - s_j)}{\prod_{j=1}^M (s - s_j)}. \quad (3.68)$$

It is clear that if the first term of the numerator on the right hand side of equation (3.68) is deleted, the numerator is reduced from an expression of order M to an expression of order $M - 1$. This would imply that one of the M load relaxation modes would have no rotational counterpart. Neglecting the first term of the numerator of equation (3.68) on the right hand side of the equation is correct for the $M - 1$ roots for which the imaginary part is orders of magnitude smaller than the real part, as outlined by Wu and Peltier (1984). For these roots, the approximation

$$\sum_{k=1}^M x'_k \prod_{j \neq k}^M (s - s_j) = 0 \quad (3.69)$$

is valid, being a purely real expression resulting in $M - 1$ real roots. These real roots constitute the $M - 1$ rotational inverse relaxation times associated with all modes except the $M0$ mode.

For a root which has a large imaginary value, comparative in strength with the variables x'_j , an argument that the first term of the numerator on the right hand side of (3.68) is negligible with respect to the second term of this numerator is no longer valid. In fact, such a complex-valued root leads to a real part, i.e. the $M0$ rotational relaxation mode, which has the same order of magnitude as the other (real) $M - 1$ roots.

This complex-valued mode, with the Chandler wobble frequency for a stratified Earth as imaginary part, has thus to be derived from the complete complex-valued equation

$$\prod_{j=1}^M (s - s_j) + i \sum_{k=1}^M x'_k \prod_{j \neq k}^M (s - s_j) = 0. \quad (3.70)$$

A highly-accurate analytical approximation formula can be derived from (3.70) for the $M0$ rotational relaxation mode. This approximation facilitates polar wander simulations in earth models with a large number of layers. Indeed, it is well known that complex rootfinding procedures are numerically more difficult to apply and are less reliable than real-valued rootfinding procedures. When an analytical formula can be obtained for the only complex root that has a non-negligible imaginary part, one can employ rootfinding procedures for real numbers using equation (3.69) instead of complex numbers using (3.70) in TPW calculations.

Before deriving this formula, an example concerning the foregoing remarks might be illustrative. Table 3.3 gives the values for the densities and rigidities of a 5-layer model. The mantle has a uniform viscosity of 10^{21} Pa s. In Table 3.4 the 9 inverse load relaxation times s_j and the 9 rotational relaxation roots a_j are given for this 5-layer earth model. The complex rotational relaxation roots a_j are determined by applying a complex rootfinding procedure to (3.70). It is

Table 3.3. Density and rigidity values for the 5-layer model

r (km)	ρ (kg/m ³)	μ (N/m ²)
6371 - 6250	3184	6.02×10^{10}
6250 - 5951	3434	7.27×10^{10}
5951 - 5701	3857	1.06×10^{11}
5701 - 3480	4878	2.19×10^{11}
3480 - 0	10932	0

clear from Table 3.4 that 8 roots have imaginary values which are negligible in strength, and that 1 root has a large imaginary value. This large imaginary value represents the Chandler wobble. The real value of this root is the $M0$ rotational relaxation mode. This mode is not negligible at all; on the contrary, it is often the strongest relaxation mode, as illustrated by the last two columns in Table 3.4. In Table 4 of Wu and Peltier (1984), where 9 load relaxation and 8 rotational relaxation roots are given, it is this $M0$ rotational relaxation mode which is lacking. The small imaginary values of the other rotational relaxation roots in Table 3.4 indicate that these modes are also characterized by wobbles. One can easily prove from (3.70) that the imaginary parts of these modes are not equal to zero, thus excluding the possibility that these imaginary values are the result of numerical inaccuracies. This is interesting from a physical point of view, although the amplitudes of these wobbles are too small to be of any immediate significance.

Filtering the Chandler wobble out after the rotational relaxation roots have been found is not the same as deleting the mode in which the Chandler wobble will appear before (3.70) is solved. One might thus think that the omission of the $M0$ rotational relaxation mode has marked consequences for the TPW simulations. However, equation (79) in Wu and Peltier (1984) contains the extra term $D_1 f(t)$ inside the square brackets, which is only created when the number of rotational relaxation modes is one less than the number of load relaxation modes. It will be shown that this term contains approximately the same contribution as is found from the relaxation of the $M0$ rotational relaxation mode after the Chandler wobble is filtered out. This equivalence will be pointed out and discussed after the analytical formula for the $M0$ rotational relaxation mode is derived.

4.3 ANALYTICAL FORMULA FOR THE $M0$ ROTATION MODE

As shown in the last section, the numerator on the right hand side of equation (3.68) has $M - 1$ solutions $s = a_j$ ($j = 1, \dots, M - 1$) for which the imaginary part can be neglected. These $M - 1$ solutions can be found by applying a real-

Table 3.4. Inverse load relaxation times s_j , inverse rotational relaxation times a_j , load Love numbers k_j^L and tidal-effective Love numbers k_j for the 9 relaxation modes of the 5-layer model of Table 3.3, with a uniform mantle viscosity of 10^{21} Pa s. The labeling of the modes agrees with the labeling in Table 4 of Wu and Peltier (1984). Note that the M0 mode is the strongest mode for both load relaxation and tidal-effective relaxation

mode	$-s_j$ (kyr ⁻¹)	$-a_j$ (kyr ⁻¹)	k_j^L (kyr ⁻¹)	k_j (kyr ⁻¹)
M2	2.90×10^{-5}	$7.32 \times 10^{-5} - 9.64 \times 10^{-10} i$	-1.45×10^{-7}	7.17×10^{-9}
M1	1.29×10^{-3}	$1.06 \times 10^{-2} - 1.73 \times 10^{-6} i$	-7.01×10^{-5}	7.72×10^{-6}
L0	1.09×10^{-1}	$1.19 \times 10^{-1} - 1.89 \times 10^{-6} i$	-1.69×10^{-4}	7.84×10^{-4}
C0	4.50×10^{-1}	$1.02 - 1.50 \times 10^{-4} i$	-1.41×10^{-1}	8.08×10^{-2}
M0	2.02	$1.95 - 5.10 \times 10^3 i$	-3.32×10^{-1}	4.21×10^{-1}
T1	2.48	$2.38 - 2.19 \times 10^{-5} i$	-7.82×10^{-2}	8.13×10^{-2}
T2	2.84	$2.62 - 2.45 \times 10^{-5} i$	-2.80×10^{-1}	3.73×10^{-1}
T3	3.56	$3.47 - 2.60 \times 10^{-5} i$	-9.95×10^{-2}	1.13×10^{-1}
T4	4.00	$3.89 - 2.94 \times 10^{-5} i$	-1.15×10^{-1}	1.79×10^{-1}

valued rootfinder procedure to (3.69). The root which contains the Chandler wobble as imaginary part and the M0 rotational relaxation mode as real part must be solved from equation (3.70). If we split this root $s = a_{M0}$ into its real and imaginary parts as $a_{M0} = a_R + ia_I$, then it is clear from Table 3.4 that $|a_I| \gg |a_R|$ and $|a_I| \gg |s_k|$ hold for all M load relaxation modes k .

With this, the first term of equation (3.70) can be approximated by

$$\prod_{j=1}^M (s - s_j) \approx i^M a_I^M + i^{M-1} \sum_{k=1}^M (a_R - s_k) a_I^{M-1}, \quad (3.71)$$

while the second term of (3.70) can be approximated by

$$i \sum_{k=1}^M x'_k \prod_{j \neq k}^M (s - s_j) \approx i^M \sum_{k=1}^M x'_k a_I^{M-1} + i^{M-1} \sum_{k=1}^M x'_k \sum_{j \neq k}^M (a_R - s_j) a_I^{M-2}. \quad (3.72)$$

The sum of equations (3.71) and (3.72) has terms which all contain either i^M or i^{M-1} . Irrespective of the value of M , the terms must thus obey the relations

$$a_I^M + \sum_{k=1}^M x'_k a_I^{M-1} \approx 0 \quad (3.73)$$

and

$$\sum_{k=1}^M (a_R - s_k) a_I^{M-1} + \sum_{k=1}^M x'_k \sum_{j \neq k}^M (a_R - s_j) a_I^{M-2} \approx 0. \quad (3.74)$$

Equation (3.73) yields the expression for the imaginary part of the root as

$$a_I \approx - \sum_{k=1}^M x'_k \quad (3.75)$$

while equation (3.75) in (3.74) leads to

$$- \left(\sum_{k=1}^M (a_R - s_k) \right) \left(\sum_{k=1}^M x'_k \right) + \sum_{k=1}^M x'_k \sum_{j \neq k}^M (a_R - s_j) \approx 0, \quad (3.76)$$

and this can be reduced to

$$\sum_{k=1}^M x'_k (a_R - s_k) \approx 0. \quad (3.77)$$

From equation (3.75) and (3.77) we thus have as M -th complex-valued root of equation (3.70)

$$a_{M0} \approx \left(\sum_{k=1}^M x'_k s_k / \sum_{k=1}^M x'_k \right) - i \sum_{k=1}^M x'_k. \quad (3.78)$$

The real part of this root gives the rotational inverse relaxation time of the fundamental mantle mode $M0$, while the imaginary part gives the Chandler wobble frequency σ_0 of the stratified model (compare also with Wu and Peltier's (1984) equation (68)):

$$\sigma_0 \approx \sum_{k=1}^M x'_k \quad (3.79)$$

or

$$\sigma_0 \approx \sigma_r \sum_{k=1}^M \frac{k_j^T}{k_f^T s_j}. \quad (3.80)$$

For the 5-layer model in Table 3.3 the complex root as given by (3.78) has the value

$$a_{M0} = -1.9467033513 + 5096.94178i \text{ kyr}^{-1}, \quad (3.81)$$

while from a complex rootfinder applied to (3.70) with quadruple precision the root is determined as having the value

$$a_{M0} = -1.9467033558 + 5096.94204i \text{ kyr}^{-1}. \quad (3.82)$$

The analytical formula (3.78) thus gives an extremely accurate approximation of a_{M0} .

4.4 UNIFICATION OF THE DIFFERENT APPROACHES

Equations (3.78) and (3.80) are not only useful in the model approach following the method of Sabadini *et al.* (1982a), but also prove helpful in establishing the equivalence with the model approach in Wu and Peltier (1984) for secular TPW.

If we consider equation (79) of Wu and Peltier (1984), then the formulation of Sabadini *et al.* (1982a) alters this equation by the following four points:

(1) The term D_1 in Wu and Peltier's (1984) equation (79) becomes zero. This term D_1 is an elastic term which arises from the first term (being 1) on the right hand side of Wu and Peltier's (1984) equation (75). The term D_1 is a direct consequence of the fact that there is one rotational relaxation mode less than the number of load relaxation modes. This term of unity would be absent if there were N rotational relaxation modes corresponding with the N load relaxation modes inside the square brackets of the last line in Wu and Peltier's (1984) equation (74).

(2) The term D_2 becomes $-i\sigma_0 D_2$ in Wu and Peltier's (1984) equation (79), but at the same time the term D_2 is divided by the extra root $-a_{M0}$ of our equation (3.78) in Wu and Peltier's (1984) equation (80). As the imaginary part of equation (13) is orders of magnitude larger than the real part, the effect on Wu and Peltier's (1984) equation (80) is that D_2 is to a high approximation divided by $-i\sigma_0$, so that the total effect on Wu and Peltier's (1984) equation (79) is that the original term D_2 remains unchanged (note that our roots a_i have the opposite sign of Wu and Peltier's (1984) corresponding roots λ_i).

(3) The terms E_i in Wu and Peltier's (1984) equation (79) become $-i\sigma_0 E_i$, but at the same time the terms E_i in Wu and Peltier's (1984) equation (80) are to a high approximation divided by the extra term $-i\sigma_0$, which is again a consequence of the fact that the extra rotational relaxation mode has an imaginary term in our equation (3.78), being orders of magnitude larger than the real part. The net effect is thus, just as in point (2), that the terms E_i remain to a high approximation unchanged in Wu and Peltier's (1984) equation (79).

(4) The final change in Wu and Peltier's (1984) equation (79) concerns the addition of the extra term $E_N f * \exp(a_{M0} t)$. This term causes the wobble. It turns out that if this extra term is averaged over time, i.e. when the wobble is filtered out, then the contribution which remains is numerically equal to D_1 to a high approximation.

The net effect of points (1)-(4) is thus that the elastic term D_1 in Wu and Peltier's (1984) theory contains the signal that is contributed by Sabadini *et al.*'s (1982a) M0 rotational relaxation mode when the Chandler wobble is filtered out. Together with points (2) and (3), one thus can conclude that to a high

approximation the theoretical developments in Sabadini *et al.* (1982a) and Wu and Peltier (1984) lead to the same results for secular TPW simulations.

A corollary of this result is that the discrepancy in the glacially induced TPW results between Peltier and Jiang (1996) and other authors, which was briefly discussed in the introduction, is due to the application of the theory. A first step to unify the approaches by Sabadini *et al.* (1982a) and Wu and Peltier (1984) was taken in Sabadini *et al.* (1984b).

5. LONG-TERM BEHAVIOR OF THE ROTATION EQUATION

The rotation equation (3.57) is characterized by terms of different nature, the first one with the pole in $s = 0$ and the second one consisting of a summation over a series of poles in $s = a_i$. Independently of the strength of these two contributions, the physics of these two contributions is quite different. Once multiplied in the Laplace domain by $1/s$, denoting a constant load in time, the term in A_0 provides a polar wander which grows linearly in time, while the terms in a_i excite exponentially decaying polar shifts. The impact of the different nature of these terms on polar wander studies has not been elucidated in the literature, mainly because interest has been focussed on the present day rotational response of the Earth to the last glacial cycle of the Pleistocene deglaciation, which arises solely from the exponential terms. This section is thus devoted to the understanding of the physics underlying these two different classes of s -poles.

It is possible to get a deep insight into the physics of the $s = 0$ pole, whose strength is given by A_0 , in terms of normal mode theory by studying the long-term behavior of the equation for retrieving $\mathbf{m}(s)$. In equation (3.44) we can take the limit for $|s| \ll |s_i|$, which means that the Earth's rotation is studied for time scales larger than the characteristic relaxation times of the normal modes. In this equation, the terms $k_i/(s - s_i)$ can be expanded to first order

$$\frac{k_i}{(s - s_i)} \sim -\frac{k_i}{s_i} \left(1 + \frac{s}{s_i}\right). \quad (3.83)$$

By taking into account the expression of the fluid tidal Love number

$$k_f = k_e - \sum_{i=1}^M \frac{k_i}{s_i} \quad (3.84)$$

and the first order approximation of the relaxing terms above, the rotation equation takes the form

$$\mathbf{m}(s) = \frac{-i\sigma_r \Phi_{\mathbf{L}}(s)}{s \left(1 - \frac{i\sigma_r T_1}{k_f}\right)}, \quad (3.85)$$

with T_1 denoting the time scale of readjustment of the equatorial bulge given by

$$T_1 = \sum_{i=1}^N \frac{k_i}{s_i^2}. \quad (3.86)$$

Comparison between equations (3.85) and (3.56) indicates that the explicit expression of the term A_0 is given by

$$A_0 = \frac{1}{(1 - \frac{i\sigma_r}{k_f} T_1)}, \quad (3.87)$$

since the $s = 0$ limit of the two equations must coincide. Equation (3.85) becomes in the time domain

$$\frac{d}{dt} \mathbf{m} = \frac{-i\sigma_r \Phi_{\mathbf{L}}}{(1 - \frac{i\sigma_r}{k_f} T_1)} = -i\sigma_r A_0 \Phi_{\mathbf{L}}. \quad (3.88)$$

The basic equations for true polar wander in the $s = 0$ limit can thus be cast in terms of the time scale required for the readjustment of the rotational bulge induced by perturbation in the Earth's rotation, as shown in Sabadini and Yuen (1989) or Spada *et al.* (1992b). The long time scale rotational state of the Earth depends on the rheology of the mantle through the term T_1 .

In the following Figure 3.5, the time scale T_1 for the readjustment of the equatorial bulge is plotted for a 4-layer earth model, as a function of the viscosity ratio between the lower and upper mantle, for two different lithospheric thicknesses L ($L = 0$, panel (a) and $L = 100$ km, panel (b)) and for a fully adiabatic and chemical boundary at 670 km (solid and dashed curves).

Exercise. Taking into account the expression of the fluid tidal Love number

$$k_f = k_e - \sum_{i=1}^M \frac{k_i}{s_i} \quad (3.89)$$

and the first order approximation of the relaxing terms above, demonstrate that the long time scale version of the rotation equation takes the form in the time domain

$$\frac{d}{dt} \mathbf{m}(t) = \frac{1}{(1 + i/R_o)} \frac{\Delta I_{13} + i\Delta I_{23}}{T_1(C - A)}, \quad (3.90)$$

where the a-dimensional rotational number R_o is defined by

$$R_o = \frac{\Omega T_1 J_2}{I^*}, \quad (3.91)$$

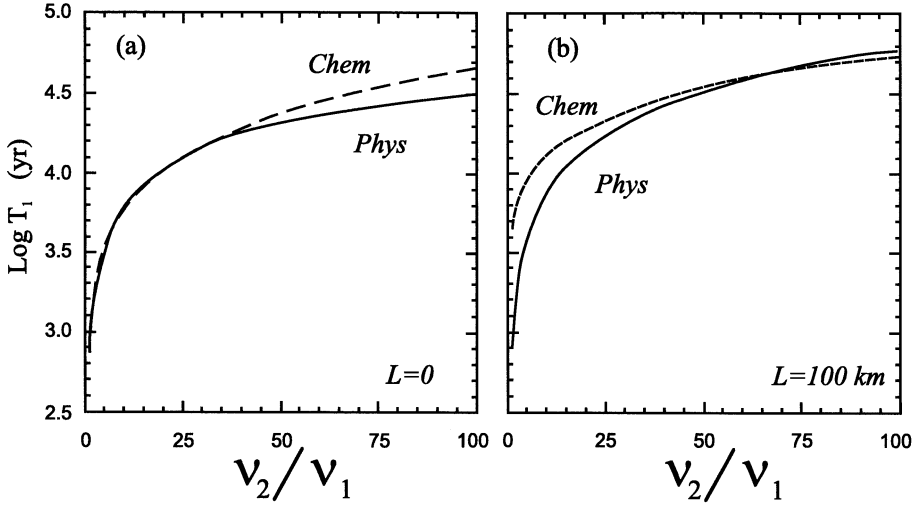


Figure 3.5. T_1 as a function of the lower mantle-upper mantle viscosity ratio ν_2/ν_1 . ν_1 is fixed at 10^{21} Pa s. The solid line corresponds to a mantle of uniform density to mimic the effects of a completely adiabatic mantle ($\xi = 1$, see Chapter 1), while the dashed curve stands for a fully non-adiabatic (chemical) 670 km discontinuity ($\xi = 0$, see Chapter 1). Panels (a) and (b) stand for a model without lithosphere (lithospheric thickness $L = 0$) and for a model carrying a lithosphere of 100 km ($L = 100$ km).

with being $I^* = A/Ma^2$, J_2 the dynamic form factor

$$J_2 = \frac{(C - A)}{Ma^2} \quad (3.92)$$

and T_1 as given by equation (3.86).

Equation (3.90) shows that true polar wander can thus be given in terms of the dimensionless parameter R_o and contains in a simple, clear fashion the physics underlying the long-term rotation behavior. This number depends on the separation of the two time scales $2\pi\Omega^{-1}$ and T_1 the length of the sidereal day and the span of time required for the readjustment of the rotational bulge induced by perturbation in the Earth's rotation (Sabadini and Yuen, 1989). Depending on the two limits $R_o \gg 1$ and $R_o \ll 1$, it is in fact possible, as we show in Chapter 7 dealing with the rotational behavior of the terrestrial planets, to quantify their long-term rotational behavior. Through T_1 , R_o depends on the rheology of the mantle. A precise estimate of this number for the solid planets cannot be obtained due to the large uncertainties in their rheological structure. In spite of the uncertainties due to our lack of information concerning mantle rheology, reasonable bounds for R_o can be deduced for most of the solid planets,

sufficient to describe the basic characteristics of their long-term rotation. This problem will be tackled in Chapter 7.

Chapter 4

POLAR WANDER AND $\dot{J}_2$ INDUCED BY ICE-SHEET LOADING

1. THE CONCEPT OF TRUE POLAR WANDER (TPW)

At the end of the 19th century the theoretically predicted wobbling of the Earth's rotation axis with respect to the Earth's surface was observed. Two periodic movements have in fact been discerned from astronomical observations: one with a component of one year, related to seasonal influences, and one with a component of about 14 months, which was readily associated with Euler's free nutation of a rotating deformable body. From astronomical observations, Lambert (1922) and Wanach (1927) derived that the rotation axis of the Earth, besides these two periodic components, also shows a secular (linear) displacement with respect to the Earth's surface. Lambert (1922) found from an observation time series covering the period 1900-1917 that on the Northern Hemisphere the axis moves in the direction of 90° W longitude with a speed of 1.72 degrees per million years (deg/Myr). On the basis of a somewhat longer time series (1900-1926), Wanach (1927) found a value of 1.31 deg/Myr in the direction 42° W longitude. These observations have been refined as longer time series became available and after the influence of continental drift had been recognized and was corrected for. One of the most recent values comes from McCarthy and Luzum (1996), who report a polar wander speed of 0.925 ± 0.022 deg/Myr in the direction $75.0^\circ \pm 1.1^\circ$ W longitude. To put these values in perspective: the yearly and 14-month harmonic components have amplitudes of about 10 - 15 meters; the secular component that results after filtering these two harmonic components out has a rate of about 10 centimeters per year (one degree on the Earth's surface is about 110 km).

The observed polar wander is not an artificial consequence of continental drift (the plate on which an observatory is situated shifts with respect to the other plates; the plate through which the rotation axis of the Earth pierces shifts with

respect to the deep mantle; etc.). That is, polar wander is corrected for these plate motions. Uncorrected polar wander is usually dubbed apparent polar wander (APW); corrected polar wander is usually called true polar wander (TPW). Thus, the secular shift of the rotation axis is not due to the continental drift of the particular plate which it pierces on the Northern or Southern Hemisphere, which would be called APW, but is supposed to reflect a true wander with respect to the deep mantle. It is therefore called TPW. The question which emerges immediately from this 'with respect to the deep mantle' is how one can determine the mantle reference frame. For long-term TPW, the hot-spot reference frame is usually taken for this fixed reference frame.

Hot-spots are point-like places on the Earth's surface that show a higher heat flow than average. They are often associated with large basaltic outpourings, both on continents and in oceans, moving relatively undisturbed through mid-ocean ridges. Whereas the rising material that forms the mid-ocean ridges (raised lineaments in the oceans that signify the places where the plates drift apart) is generally thought to have its origin in the shallow upper mantle, the relatively undisturbed movement of these hot spots through mid-ocean ridges indicates that the rising material of the hot spots comes from the deep mantle. Although there is some inter-hot-spot drift it remains limited to velocities of a few cm per year at most, whereas plates move with velocities up to 10-15 cm per year. Thus, a relatively stable (stable down to a few cm per year) hot-spot reference frame can be established that is thought to coincide with a hypothetical deep-mantle reference frame. Here we will loosely refer to the terms polar wander or true polar wander (TPW) with respect to this reference frame. Although the true nature of hot-spots is still a matter of debate, one may associate them with more or less stationary mantle convection patterns. This would imply that whenever a large change of the global convection pattern emerges, being accompanied by a large reorganization of the plates, then the concept of true polar wander becomes meaningless as the hot-spot reference frame is destroyed or reorganized. This limits the concept of TPW to periods of about 100 Myr, and one should be careful not to misinterpret a redistribution of hot-spots during plate reorganization periods as a "catastrophic turnover" of the rotation axis. Apart from this, there is evidence of TPW on timescales of both millions of years (Besse and Courtillot, 1991) and thousands of years (Dickman, 1977).

Nakiboglu and Lambeck (1980) and Sabadini and Peltier (1981) have shown that the present-day secular drift of the rotation axis can be explained by post-glacial rebound due to the disappearance of the Pleistocene ice sheets. Although the rate of polar shift appears to be highly dependent on mantle viscosity, thus not being discriminative with respect to pointing out the cause behind the observed shift (change the load and one can find another value for the viscosity so that the observed polar wander drift rate is matched again), the direction of the polar

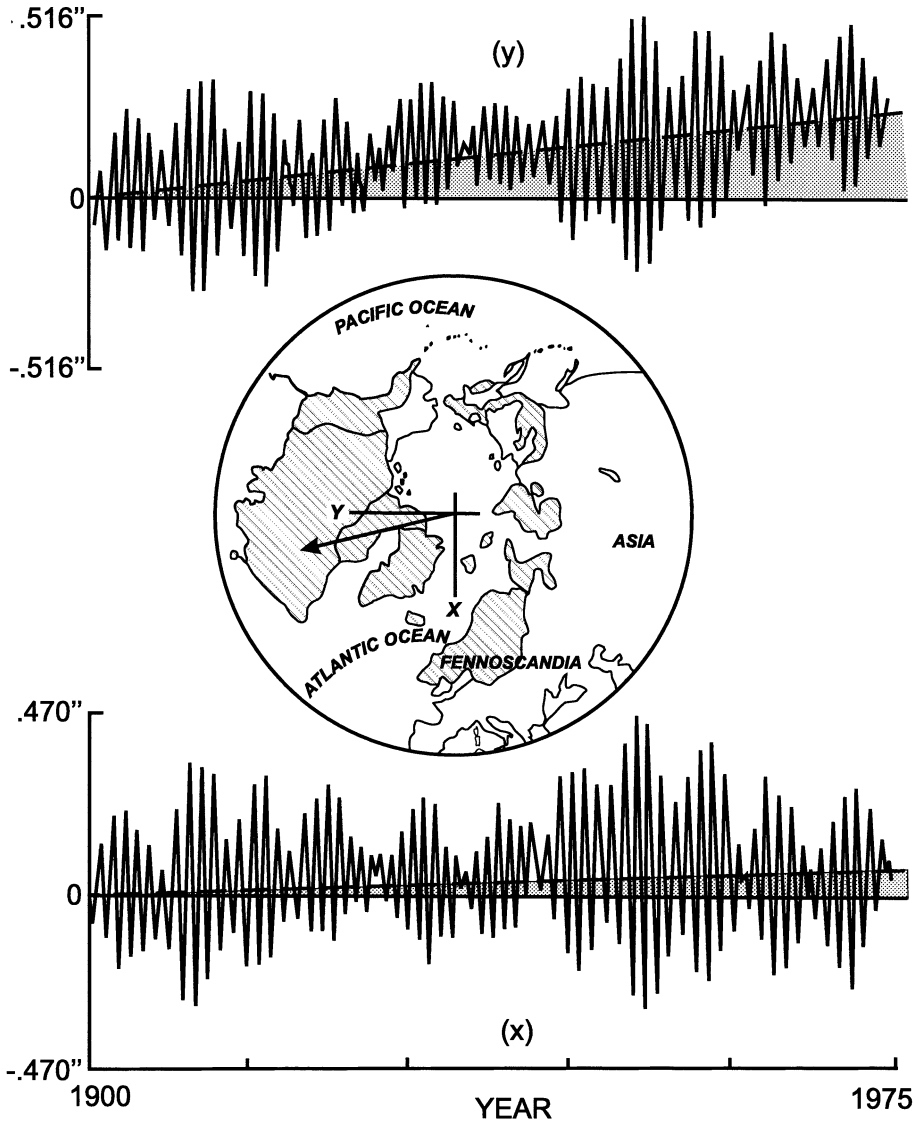


Figure 4.1. Cartoon showing the Pleistocene ice sheets and the direction of the present-day secular wander of the rotation axis. Polar motion is given in the conventional international origin (CIO) coordinates (x axis through Greenwich), on the basis of the reduction of the International Latitude Service data as described by Dickman (1977). Redrawn from Figure 1 in Sabadini and Peltier (1981).

shift is seen as a strong indication that post-glacial rebound is the main driving mechanism, as shown in Figure 4.1.

Even with the most simple ice models, for instance containing three ice disks covering Canada, Scandinavia and Antarctica and melting 10,000 years ago, the direction of the secular drift resulting from the present-day, solid-Earth rebound is quite accurately reproduced. Although there is a high correlation, it is not the same as a proof. The possibility that the high correlation might be a coincidence cannot be excluded at the moment. Indeed, a number of papers have been published in recent years that argue for additional forcings being responsible for causing the observed present-day polar drift.

On the million-year time scale, subduction (e.g., Ricard *et al.*, 1992) and mantle convection (e.g., Steinberger and O'Connell, 1997) are considered to be the major contributors to TPW, while on the thousand-year time scale the redistribution of surface loads due to the growing and melting of huge ice sheets is considered to be one of the main forcing mechanisms for TPW. Steinberger and O'Connell (1997) derive from their mantle heterogeneities advection model a contribution of 40 per cent to the observed secular TPW rate. Contemporary forcings might induce TPW rates comparable to the secular TPW velocity: some of the models in Vermeersen *et al.* (1994) on (geoid-constrained) neotectonics of the Himalayas and Tibetan Plateau and in James and Ivins (1997) on mass changes of the Greenland ice sheet induce TPW velocity rates which are comparable to the observed present-day TPW rate. Taken together, all these geophysical forcings (mantle convection, tectonics, present-day ice and water redistributions, etc.), with their associated uncertainties, make it very difficult to extract unique information about the viscosity of the Earth's mantle from TPW simulations. We will assume in this chapter, however, that polar wander during the past one million years has been caused mainly by the waxing and waning of great ice sheets.

In parallel with the analysis of a TPW driven by post-glacial rebound, this chapter will also deal with an analysis of the degree two harmonic component of the geopotential perturbation, J_2 , delaying a detailed analysis of the higher harmonics, including the effects of present-day ice loss in Antarctica and Greenland, to Chapter 5.

2. THE INFERENCE OF MANTLE VISCOSITY FROM TPW AND J_2 DATA

The inference of the viscosity structure of the mantle is a basic issue in geodynamics. A variety of important geodynamical processes, from the long time scale convection in the mantle to the faster response of the Earth to Pleistocene deglaciation, depend on this parameter.

The viscosity of the Earth plays a major role in the connection between changes in surface load and changes in Earth rotation. If the viscosity of the Earth is high, deformation does not proceed so readily as if the viscosity is low. In the case of the connection between changes in surface load and induced po-

lar wander, there are two relaxation processes: load relaxation and centrifugal relaxation. The interplay between these two relaxation processes, elucidated in Chapter 3, is what makes the dependence on viscosity somewhat complicated but certainly interesting to be analyzed. In spite of the efforts of several authors in the last decades, a general consensus on the viscosity profile of the mantle has not been reached yet. The use of two different classes of observables in modeling mantle deformation processes, convection and postglacial rebound, which occur on different time scales, is certainly one of the major causes for the discrepancies among the various inferences of mantle viscosity. It has been suggested that the creep properties of the mantle do not remain constant at short and long time scales. Convection and postglacial rebound may be controlled by steady-state and transient creep, respectively. If this is the case, modeling of postglacial rebound by means of a steady-state rheology causes a bias in the inference of viscosity (Sabadini *et al.*, 1985; Peltier, 1985). This point of view has been weakened recently by rebound analyses showing that a number of glacial isostatic adjustment data are consistent with a conspicuous viscosity increase in the lower mantle (Nakada and Lambeck, 1989) that is generally found from long-wavelength geoid analyses (Richards and Hager, 1984). Papers by Forte and Mitrovica (1996) and Mitrovica and Forte (1997) reinforce this indication by means of a joint inversion of mantle convection and postglacial rebound data that require a significant viscosity increase with depth in the mantle. Estimated viscosity profiles differ substantially from one another, even within the various postglacial rebound analyses. Typically, some studies predict a rather uniform mantle viscosity (Yuen *et al.*, 1986; Tushingham and Peltier, 1992; Spada *et al.*, 1992b), while other analyses require a substantial viscosity contrast at the interface between the upper and lower mantle (Lambeck *et al.*, 1990). Mitrovica (1996), to whom we refer for an exhaustive discussion of these issues, has provided a possible explanation for these apparently contradictory results. He suggests that they are consequences of a misinterpretation of the Haskell (1935) value, considered as limited to the upper mantle after the results of modern seismology. Other causes of apparent inconsistencies among the various glacial isostatic studies may be ascribed to the use of different Earth models, different observables considered by the various authors, and insufficient search in the parameter space of the rebound models (Lambeck *et al.*, 1996).

Other uncertainties that accompany deriving radial viscosity profiles from rotational studies are the neglect of lateral viscosity variations and uncertainties in the load history of the Pleistocene ice models. There are reasons, however, that these will affect the results to a lesser extent. Although large lateral variations in viscosity are very likely to be present in the upper mantle, their influence on second-degree harmonics is found to be small (D'Agostino *et al.*, 1997). For low-degree phenomena like Earth rotation and global geoid change we thus expect that volume averaging over lateral viscosity variations in the upper mantle,

and using this value in our models, will give about the same results as when models are used in which these lateral viscosity variations are retained. To put this in another way, it might be very difficult to obtain information on lateral viscosity variations from degree 2 rotational data, just as it is also difficult to obtain detailed information on the radial viscosity profile for a large number of layers from studying harmonic degree 2 phenomena only. Indeed, we have already seen in Chapter 2 that for harmonic degree 2, saturated continuum limits are reached for the Love numbers for models with a small number of layers largely independent of the form of the radial viscosity profile. Considering this, and considering the uncertainties with respect to the forcing mechanisms, we have decided to take only two viscosity layers in the models shown in Chapter 2, whereas the elastic parameters and density that we use (PREM) are more finely graded. From earlier studies (e.g., Wu and Peltier, 1984), it is known that the last ice age has the most impact on present-day rotational changes, thereby diminishing the influence of uncertainties in the history of former Pleistocene ice cycles.

The observational data considered in this section are taken from Dickman (1977) as shown in Figure 4.1 and McCarthy and Luzum (1996) for TPW, and from Yoder *et al.* (1983) and Devoti *et al.* (1997) for $\dot{J}_2$. For a review on how these data have been derived see the introduction to the paper by Peltier and Jiang (1996).

3. LOADING

From oxygen isotope analysis of ocean sediments (e.g., Shackleton and Opdyke, 1976), it has been deduced that the great ice sheets had a growth period of about 90,000 years and a decay period of about 10,000 years, so a total period of about 100,000 years for one complete cycle (water contains the isotopes ^{16}O and ^{18}O ; depending on the amount of ice that has accumulated on land, the relative abundance of the two isotopes which are incorporated in ocean sediments will vary due to their specific mass difference: the lighter isotope evaporates more easily from the oceans, giving an over-abundance of this isotope in ice with respect to ocean water, thus creating an over-abundance of the heavier isotope in the oceans during glacial times, which is recorded in the sediments). The period of 100,000 years is equal to the time in which the ellipticity of the orbit in the Earth around the Sun changes (one of the periods in the so-called Milankovitz cycle of harmonic variations of the Earth's orbit around the Sun), but whether there is a direct connection between ellipticity change and ice age cycles is a contentious issue at the moment. Many researchers accept the theory that the change in ellipticity can trigger and end an ice age cycle indirectly, that is, the change in solar insolation is not considered powerful enough to induce a glacial period directly but it can trigger non-linear changes in ocean currents, which can be the driving agent of the climate change. However, more

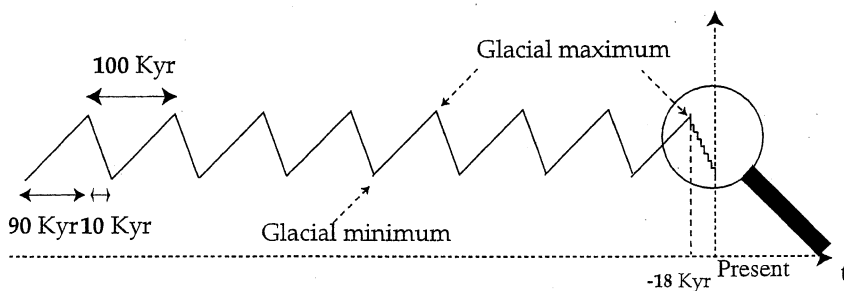


Figure 4.2. Pleistocene glaciation and deglaciation history as used in the models. The history contains seven glacial ramp-shaped cycles followed by a linear ice accumulation period and a deglaciation period consisting of a set of discrete unloading steps (after ICE-3G (Tushingham and Peltier, 1991)). Each cycle has a 90 kyr glaciatic period. The seven glacial precycles have deglaciation periods of 10 kyr. The final glacial cycle, which starts 18 kyr before present, has a 13 kyr deglaciation period of Heavyside loadings. The ice masses are assumed to remain constant after 5 kyr before present. The maximum amount of ice accumulated at the end of a glaciatic period is the same as ICE-3G gives at 18 kyr before present. The minimum amount of ice at the end of a deglaciation period is the same as ICE-3G gives at 5 kyr before present (Figure 2 in Vermeersen *et al.*, 1997).

recently, other theories have emerged, such as that in which changes in interplanetary dust density with a period of 100,000 years as a result of changes in orbital inclination might be the prime mover (Muller and MacDonald, 1995; Farley and Patterson, 1995). Whatever the cause, oxygen isotope analyses have discerned at least eight glacial cycles during the Pleistocene.

Figure 4.2 shows a cartoon of the ice load model that will be used to study the influence of ice age cycles on polar wander (Vermeersen *et al.*, 1997; Vermeersen and Sabadini, 1999).

In order to make use of equation (3.61) and of the ICE-3G model by Tushingham and Peltier (1991), we must specify the Laplace transform of the loading history $\tilde{f}(s)$. The saw-tooth function appropriate for describing the n -th cycle of the load is given, in time domain, by

$$f(t) = \begin{cases} \frac{t-a(n-1)}{a-b} & \text{if } a(n-1) < t < a(n-1) + b \\ \frac{a(n-1)+b-t}{a-b} & a(n-1) + b < t < an. \end{cases}$$

In the Laplace domain the load time history becomes

$$\tilde{f}(s) = \sum_{n=1}^8 \frac{e^{-a(n-1)s} - e^{-a(n-1)s-bs}}{bs^2} - \frac{e^{-a(n-1)s-bs} - e^{-ans}}{(a-b)s^2}, \quad (4.1)$$

where 8 cycles of glaciation-deglaciation are considered, with the $a = 100$ kyr denoting the glaciation phase and $b = 10$ kyr the deglaciation phase.

Inserting equation (4.1) into equation (3.61) and transforming back into the time domain, we obtain first that the contribution arising from the secular term A_0^* for each cycle, after the end of the last glacial event, is given by

$$\mathbf{m}(t)_{sec} = -i\sigma_r \frac{\Delta I_{13}^R + i\Delta I_{23}^R}{C - A} \frac{A_0^* a}{2}. \quad (4.2)$$

Thus, for times after the termination of ice ages, the displacement of the axis of rotation arising from the A_0^* term given by equation (3.64) remains constant to the net shift acquired during the active phases. This result is relevant for the interpretation of TPW driven by the continuous occurrence of ice ages, as shown afterwards.

The contribution arising from the a_i poles becomes appropriate for quantifying the present-day polar wander, after the 8-th cycle

$$\begin{aligned} \mathbf{m}(t) = & \sum_{n=1}^8 \left(\sum_{i=1}^M \left(-i\sigma_r \frac{\Delta I_{13}^R + i\Delta I_{23}^R}{C - A} \right) \times \right. \\ & \left[\frac{\gamma_i}{ba_i^2} \{ e^{a_i[t-a(n-1)]} - e^{a_i[t-a(n-1)-b]} \} \right. \\ & \left. \left. - \frac{\gamma_i}{(a-b)a_i^2} \{ e^{a_i[t-a(n-1)-b]} - e^{a_i(t-an)} \} \right] \right), \end{aligned} \quad (4.3)$$

where summation over n corresponds to the eight glacial cycles considered in the modeling. From this expression for $\mathbf{m}(t)$ we obtain $\lim_{t \rightarrow \infty} \mathbf{m}(t) = 0$, which shows that for long time scales there is no net polar wander associated with viscoelastic relaxation modes.

ICE-3G consists of a total of 808 circular disk loads representing the Laurentide ice sheet, the Fennoscandian ice sheet, the Northern part of Russia, Antarctica, and the southwest of South America. For all these disks, ICE-3G gives the ice decay of the last Pleistocene cycle via a set of decrements, starting 18 kyrs before present and ending 5 kyrs before the present. The Antarctic ice sheet starts to melt at a later stage than the ice sheets of the Northern Hemisphere. The Pleistocene glaciation-deglaciation history, as used in the modeling, is shown in Figure 4.2, while the total ice mass of all the disks is plotted as a function of time in Figure 4.3 for the final deglaciation phase. It might be interesting to note that there are some negligibly small changes from 5 to 3 kyrs before present; from 3 kyrs to present the ice masses of all disks remain constant.

Our eight complete glacial cycles consist of seven glacial saw-tooth precycles, while the eighth cycle consists of a linear glaciation phase which ends 18

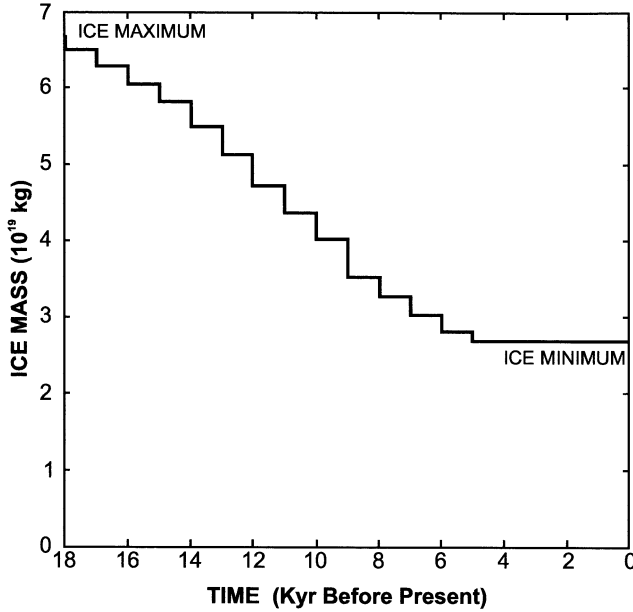


Figure 4.3. Enlargement of the final deglaciation phase in Figure 4.2. The discrete time steps of the ICE-3G model are shown. The ice mass on the vertical axis represents the ice of all sheets (Figure 3 in Vermeersen *et al.*, 1997).

kyrs before present, followed by ICE-3G. Each cycle is connected to its previous and following ones. Each of the seven precycles consists of a 90 kyr linear growth phase and a 10 kyr linear decay phase. The minimum amount of ice is the same as in ICE-3G at present, while the maximum amount of ice is the same as in ICE-3G 18 kyrs before present. Both the polar wander rates and the $\dot{J}_2$ rates are determined at present, with $\dot{J}_2$ denoting the degree 2 components of the perturbation in the geopotential.

For this temporal history the Laplace-transformed function $\tilde{f}(s)$ can be easily determined and substituted in the multidisk equivalents of the above formulations. The temporal history is simple enough, consisting of a combination of Heaviside functions and linear functions, so that performing the inverse transformations of the loading history can be done analytically, similarly to equation (4.1).

The TPW models can be relatively easily implemented in computer codes when all inverse relaxation times s_i and a_i are detected by a root-finding procedure. This is certainly the case with models which only have a limited number of layers, such as the five-layer models used in Spada *et al.* (1992b). Problems arise when not all the roots are found, as is rather the rule than the exception in models with a large number of layers. In models where only load relaxation is

important, as for $\dot{J}_2$, it does not matter if modes that have far too low a strength to be of any importance are not taken into account (or remain undetected). This is the case, for instance, of those long-term buoyancy modes which are triggered by small density contrasts between two layers as discussed in Chapter 2. However, for polar wander it is necessary to take into account all the a_i rotational counterparts of the s_i modes.

4. MANTLE VISCOSITY

This section 4.3, reworked from Vermeersen *et al.* (1997), deals with the effects of viscosity variations in the mantle on TPW and $\dot{J}_2$. The TPW results for a viscosity of 10^{21} Pa s in the upper mantle and varying lower mantle viscosity are portrayed in Figure 4.4.

In this figure, only the Laurentide, Fennoscandian, and Antarctic ice sheets are taken into account as disks, in agreement with the analyses by Yuen *et al.* (1986) and Spada *et al.* (1992b). Figure 4.4 shows, within the scheme of a five-layer model consisting of the lithosphere, three layers in the mantle and a core, the effects of constraining the average values of the density and rigidity in each layer on the basis of PREM (solid curve) or, alternatively, the effects of constraining the density and rigidity contrasts at the 420 and 670 km discontinuities in agreement with PREM (dashed curve). The latter procedure was used in the past in TPW models based on analytical schemes (Sabadini *et al.*, 1982a; Yuen *et al.*, 1986; Spada *et al.*, 1992b). The values used in the five-layer models are given in Table 2.1 for the fixed boundary contrast model and in Table 2.2 for the volume-averaged model, as discussed in Chapter 2. The two schemes predict quite different TPW signals, although the major characteristics of the two curves are the same: a general decay from high TPW value for an isoviscous mantle, the vanishing of the signal for a lower mantle viscosity of 10^{23} Pa s, and a non-monotonic behavior from 3×10^{21} Pa s to 10^{22} Pa s. The fixed boundary contrast model (dashed curve) portrays a well-developed local maximum at 10^{22} Pa s. In particular, the dashed curve crosses the observational datum of McCarthy and Luzum (1996) for a lower mantle viscosity of 1.5×10^{21} Pa s. On the basis of this result an almost homogeneous mantle was deduced from previous TPW and $\dot{J}_2$ studies (Sabadini *et al.*, 1982a; Yuen *et al.*, 1986; Spada *et al.*, 1992b). The extra buoyancy of the model with volume-averaged densities, due to the large density contrasts at 420 and 670 km depth, is responsible for an increase in the TPW signal in the whole range of lower mantle viscosities, which has the major effect of displacing the crossing of the TPW curve with the observed values to higher viscosities. As we will see below, this result has a major impact on the interpretation of the viscosity profile of the mantle based on TPW analysis, and is caused by the density contrasts as being non-adiabatic in nature, as discussed in detail hereafter in this chapter.

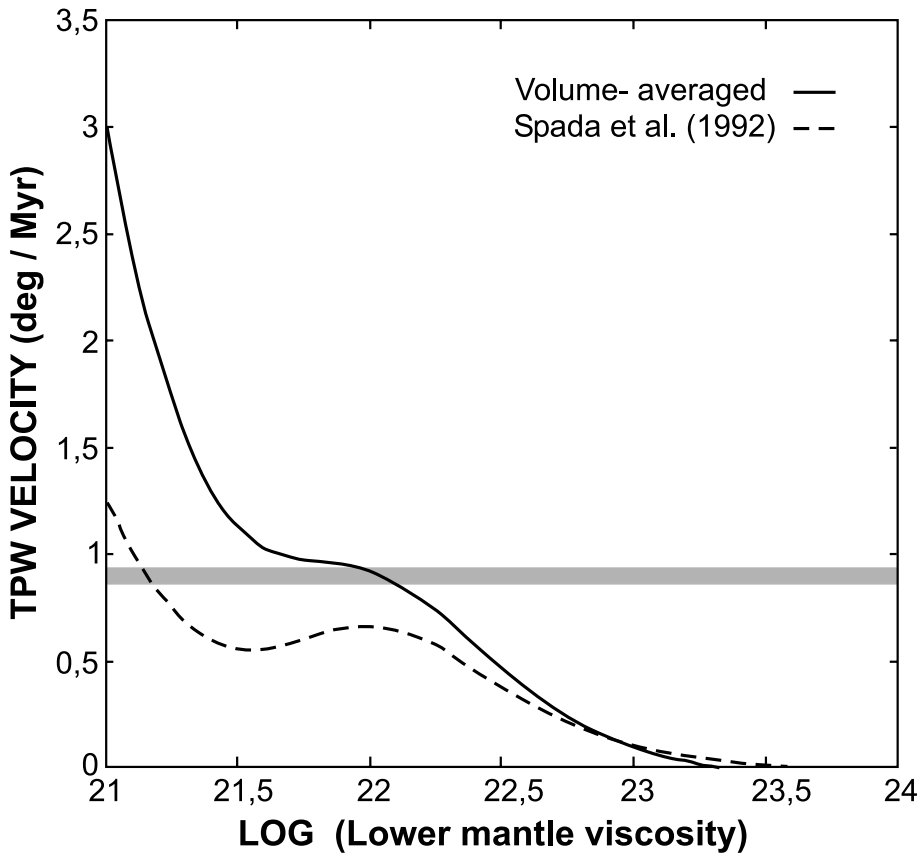


Figure 4.4. The present-day true polar wander velocity as a function of the lower mantle viscosity for two five-layer earth models. For this figure the ice load consisted of the Fennoscandian, Laurentian, and Antarctic ice sheets modeled as homogeneous disks. The value of the viscosity of the upper mantle is 10^{21} Pa s. The solid curve depicts the solution by the volume-averaged earth model in Table 2.2, and the dashed curve depicts the solution from the fixed boundary contrast earth model in Table 2.1 as used in Spada *et al.* (1992b). The hatched area depicts the observed present-day secular drift ranging between the error bars as given by McCarthy and Luzum (1996).

Figure 4.5 shows the effects of distributing the density contrasts of the mantle in a large number of layers. PREM contains 56 layers in the lithosphere and mantle. The 31-layer model in Table 4.1 is deduced from PREM in such a way that the layers take on a progressively larger thickness from the Earth's surface to the core, as done in Chapter 2.

With respect to the simpler five-layer model with volume-averaged densities, the buoyancy is now smoothly distributed over the mantle instead of being

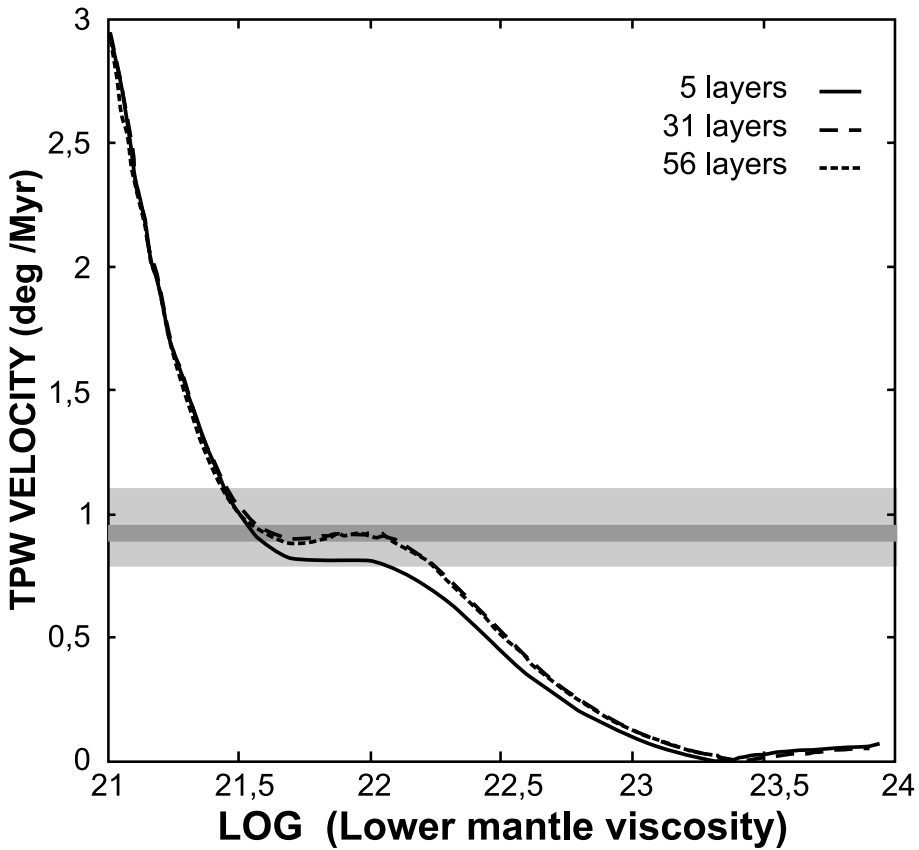


Figure 4.5. The present-day true polar wander velocity as a function of the viscosity of the lower mantle for a volume-averaged earth model. The upper mantle has a viscosity of 10^{21} Pa s. The number of layers for the three curves is indicated in the top right corner. The lightly hatched area depicts the observed present-day secular drift ranging between the error bars as given by Dickman (1977). The darkly hatched area depicts the observed present-day secular drift ranging between the error bars as given by McCarthy and Luzum (1996).

concentrated at the four major boundaries. Layering has, in fact, a major effect because it modifies the TPW curve in the proximity of the inflection of the curve at a lower mantle viscosity of 10^{22} Pa s, where crossing with the observations occurs. While for lower mantle viscosities ranging from 10^{21} Pa s to 3×10^{21} Pa s density stratification has negligible effects, PREM predicts a quite different behavior with respect to the simpler five-layer model for viscosities higher than 3×10^{21} Pa s. In fact, the TPW signal increases in the proximity of the observational data, causing a multiplicity of lower mantle viscosity solutions. These results indicate a trade-off between density and viscosity stratification in

Table 4.1. Parameters for the standard 31-layer volume-averaged Earth model derived from PREM. r is the distance with respect to the center of the Earth, ρ the density of the layer, and μ the rigidity.

layer	r (km)	ρ (kg/m ³)	μ (N/m ²)	
1	6371.0 – 6368.0	1020	2.66×10^{10}	lithosphere
2	6368.0 – 6356.0	2600	2.66×10^{10}	
3	6356.0 – 6346.6	2900	4.41×10^{10}	
4	6346.6 – 6331.0	3372	6.81×10^{10}	
5	6331.0 – 6311.0	3372	6.78×10^{10}	
6	6311.0 – 6291.0	3372	6.75×10^{10}	
7	6291.0 – 6251.0	3372	6.71×10^{10}	
8	6251.0 – 6221.0	3372	6.67×10^{10}	shallow upper mantle
9	6221.0 – 6186.0	3372	6.63×10^{10}	
10	6186.0 – 6151.0	3372	6.58×10^{10}	
11	6151.0 – 6106.0	3449	7.48×10^{10}	
12	6106.0 – 6061.0	3476	7.64×10^{10}	
13	6061.0 – 6016.0	3503	7.81×10^{10}	
14	6016.0 – 5971.0	3529	7.97×10^{10}	
15	5971.0 – 5921.0	3755	9.39×10^{10}	transition zone
16	5921.0 – 5871.0	3819	1.01×10^{11}	
17	5871.0 – 5821.0	3882	1.09×10^{11}	
18	5821.0 – 5771.0	3945	1.17×10^{11}	
19	5771.0 – 5736.0	3980	1.21×10^{11}	
20	5736.0 – 5701.0	3988	1.23×10^{11}	
21	5701.0 – 5650.0	4397	1.59×10^{11}	lower mantle
22	5650.0 – 5600.0	4423	1.68×10^{11}	
23	5600.0 – 5400.0	4501	1.78×10^{11}	
24	5400.0 – 5200.0	4620	1.91×10^{11}	
25	5200.0 – 4900.0	4759	2.06×10^{11}	
26	4900.0 – 4600.0	4921	2.24×10^{11}	
27	4600.0 – 4300.0	5078	2.41×10^{11}	
28	4300.0 – 4000.0	5205	2.58×10^{11}	
29	4000.0 – 3700.0	5379	2.74×10^{11}	
30	3700.0 – 3480.0	5509	2.89×10^{11}	
31	3480.0 – 0	10932	0	inviscid fluid core

the mantle, which will be discussed in detail at the end of this chapter, where attention is devoted to the nature of density jumps within the mantle, being due to adiabatic (or partially adiabatic) phase changes or to chemical, non-adiabatic, density contrasts. Another remarkable result of this figure is that differences between the curves corresponding to 31 and 56 layers are negligible, indicating that as far as rotational calculations are concerned, the continuous behavior

in the sense discussed in Chapter 2 is already reached with 31-layer models. Increasing the number of layers does not provide any further information on TPW.

It is interesting to compare the sensitivity of TPW as to density stratification with that portrayed by $\dot{J}_2$, defined in equation (3.66) for a simplified earth model and load. In Figure 4.6, $\dot{J}_2$ is shown for a 5-layer and 31-layer model with a fixed upper mantle viscosity of 10^{21} Pa s, varying lower mantle viscosity and realistic loading history. Density stratification has no effects on $\dot{J}_2$ between a 5-layer model and PREM once volume-averaged parameters are used. This different behavior with respect to TPW is not surprising since TPW solutions have the extra viscoelastic readjustment of the equatorial bulge above the load readjustment, which is the only one present in $\dot{J}_2$.

4.1 VARIATIONS IN DEPTH OF THE TWO-LAYER MANTLE VISCOSITY PROFILE

In this section we consider the possibility that the boundary where the viscosity contrast occurs does not coincide with the seismologically inferred base of the upper mantle. Two depths are considered, 971 and 1471 km, so both are deeper than the 670 km one shown for comparison. The 1471 km depth is close to the depth of 1400 km considered in the relative sea-level analysis by Mitrovica (1996) and the TPW and $\dot{J}_2$ studies by Peltier and Jiang (1996), while 971 km is an intermediate value.

For the upper layer viscosity fixed at 10^{21} Pa s, Figure 4.7 portrays the TPW curves for the three different depths as a function of lower layer viscosity. Increasing the viscosity in the lower layer has the effect of reducing the TPW from about 3° Myr^{-1} for a viscosity of 10^{21} Pa s to zero. Although the shape of the curves is the same for the three different depths, the deepening of the viscosity contrast has the major effect of diminishing the local maximum at 10^{22} Pa s. For 1471 km this local maximum disappears and is barely visible for 971 km. The modification in the shape of the curves has the important consequence of reducing the ambiguity in the inference of the lowermost viscosity from the admissible range of $2.8 \times 10^{21} - 1.5 \times 10^{22}$ Pa s for 670 km to the narrower range of $3 - 5 \times 10^{21}$ Pa s for 1471 km when Dickman's (1977) data are considered. The most recent TPW data by McCarthy and Luzum (1996) require a value of 4×10^{21} Pa s, being intermediate between the two previous estimates. Notice that, irrespective of the depth of the boundary, the same value of TPW is retrieved when the lowermost viscosity equals the viscosity of the upper layer, as expected. The results shown in Figure 4.7 are in distinct contrast with those portrayed by Peltier and Jiang (1996) in their Figures 7c and 9a. The major difference with respect to our results regards the drastic modification of the TPW curves when the viscosity contrast is moved from 670 to 1470 km. At first glance the TPW curve in Peltier and Jiang (1996; Figure 9a) corresponding

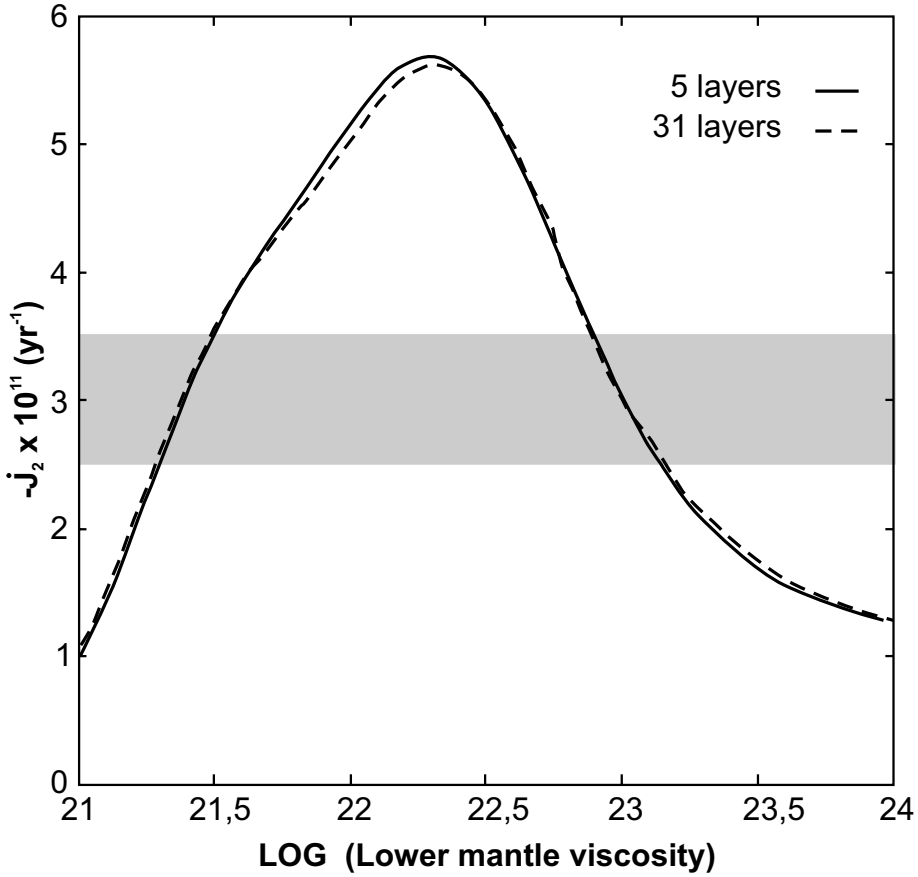


Figure 4.6. Present-day change in the non-tidal dynamic form factor $\dot{J}_2$ as a function of lower mantle viscosity and upper mantle viscosity of 10^{21} Pa s. The solid and dashed curves correspond to volume-averaged models of 5 and 31 layers. The hatched area depicts the observed present-day $\dot{J}_2$ ranging between $(-2.5 \pm 0.7 \times 10^{-11})$ 1/yr, corresponding to the solution CGS96 from the Center of Space Geodesy in Matera (Italian Space Agency), derived from LAGEOS I and LAGEOS II monthly estimates for the time interval 1985-1996 (Devoti *et al.* 1997), providing the lower bound, and from the estimate obtained by Yoder *et al.* (1983).

to the case of 1400 km, is displaced to the right with respect to their Figure 7c. This displacement is toward higher viscosities, which seems to prevent the same TPW being reproduced when both the upper and lower layers have the same viscosity of 10^{21} Pa s. This equality should be the case when the TPW curve in their Figure 9a is continued to 10^{21} Pa s. TPW for this viscosity value is not shown in their Figure 9a, but because of the behavior of the curve, which shows a monotonic increase when the lower mantle viscosity is varied

to lower values, it is difficult to imagine that the value of $1.3^\circ \text{ Myr}^{-1}$ in Figure 7c of Peltier and Jiang (1996), corresponding to 10^{21} Pa s , could be reproduced unless a maximum is located somewhere between 10^{21} Pa s and 10^{22} Pa s . Another major difference in comparison to our results is in the vanishing of TPW in Figure 7c of Peltier and Jiang (1996) for a depth of the boundary at 670 km and a lower mantle viscosity of about 10^{22} Pa s . We obtain, on the contrary, a local maximum of about $0.9^\circ \text{ Myr}^{-1}$. It should be emphasized that a non-monotonic decrease in TPW, as shown by Yuen and Sabadini (1984) and Spada *et al.* (1992b), which is consistent with the results shown in the present paper with a local maximum in proximity of 10^{22} Pa s , is also obtained in a recent analysis by Milne and Mitrovia (1996) based on compressible models.

Figure 4.8 deals with a similar analysis for $\dot{J}_2$. The most apparent effects of the lowering of the viscosity contrast are the disappearance of the high-viscosity solution branch of $2 \times 10^{23} \text{ Pa s}$ for 1471 km in the right portion of the figure and the smooth increase, from $1.5 - 2.8 \times 10^{21}$ to $2.5 - 5 \times 10^{21} \text{ Pa s}$, of the low-viscosity branch. The increase in the $\dot{J}_2$ signal for lowermost mantle viscosities higher than $3 \times 10^{21} \text{ Pa s}$ when the viscosity contrast is located deeper than 670 km agrees with the results of Mitrovia and Peltier (1993). This behavior in the $\dot{J}_2$ curve has the important implication of allowing for a single solution in the lowermost viscosity and, when comparison is made with the TPW curve for 1471 km in Figure 4.7, allowing for the simultaneous fit of the TPW and $\dot{J}_2$ data with a viscosity in the lower layer of $3 - 5 \times 10^{21} \text{ Pa s}$ when the TPW data by Dickman (1977) and $4 \times 10^{21} \text{ Pa s}$ with the latest TPW data (McCarthy and Luzum, 1996) are considered. For 670 km it is not possible to obtain a simultaneous fit of both TPW and $\dot{J}_2$ for any value of the lower mantle viscosity when the new TPW data are considered. With Dickman's (1977) TPW data the two viscosity solutions barely overlap at $2.5 \times 10^{21} \text{ Pa s}$, degrading the simultaneous TPW and $\dot{J}_2$ fit for the depth of the viscosity contrast at 670 km with respect to 1471 km. If the same comparison between our $\dot{J}_2$ analysis and Figure 9b in Peltier and Jiang (1996) is performed, the conclusions are similar to the ones in the previous figure. In Peltier and Jiang (1996, Figure 9b) the $\dot{J}_2$ curve appears to be displaced to the right with respect to their Figure 8b, again prohibiting the same $\dot{J}_2$ value from being reached when the mantle viscosity is uniformly fixed at 10^{21} Pa s in both layers and thus reproducing the same inconsistency addressed above. As for the case of TPW, Peltier and Jiang (1996) thus predict a higher viscosity in the lowermost layer of 10^{22} Pa s . The different findings of the present analysis with respect to Peltier and Jiang's (1996) cannot be due to different formulations, as shown in Chapter 3, where the equivalence of the two procedures in solving the rotational Liouville equations was demonstrated. Neither are (small) differences in the ice models, as low-degree phenomena like TPW and $\dot{J}_2$ are relatively insensitive to this (high harmonic degrees are, on the other hand).

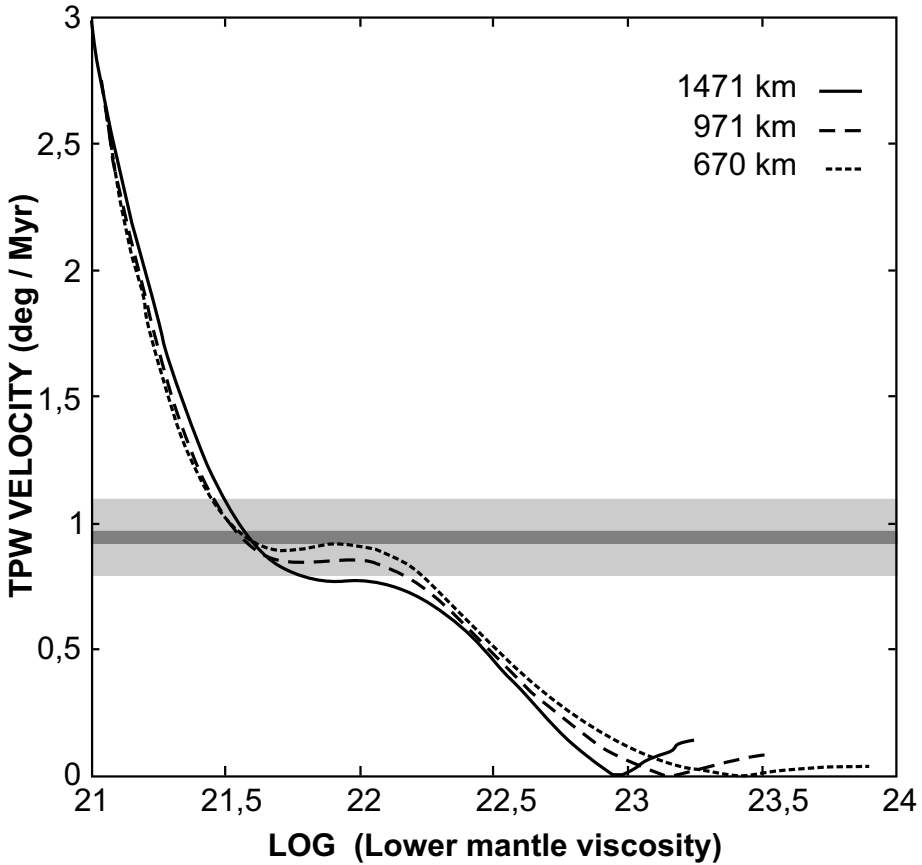


Figure 4.7. Effects of the depth of the viscosity contrast on present-day TWP as a function of the value of the viscosity of the lower mantle. The standard 31-layer model in Table 4.1 is used as the earth model. The three cases depict the results for the boundary between the upper and lower mantles (upper and lower mantle defined with respect to viscosity) at 1471, 971 and 670 km depths. The upper mantle has a viscosity of 10^{21} Pa s. The hatching has the same meaning as in Figure 4.5.

The important conclusion that we can draw from the results in Figures 4.7 and 4.8 is that when self-consistent, viscoelastic, stratified earth models are considered, a viscosity of 10^{21} Pa s cannot be limited to the seismically defined upper mantle but must be considered appropriate for a wider region of the mantle to depths of about 1400 km. It should be emphasized that this conclusion is strongly supported by the new TPW data (McCarthy and Luzum, 1996), which definitively rule out a viscosity of 10^{21} Pa s only for the upper mantle, while the TPW data by Dickman (1977) only show an indication in this sense because of the larger error bounds. This finding is only apparently inconsistent with

previous analyses based on incompressible rotational deformation models, as in Sabadini *et al.* (1982a), Yuen and Sabadini (1984), and Spada *et al.* (1992b). In these references a simultaneous fit of TPW and $\dot{J}_2$ data was found from an assumed value for the viscosity of the upper mantle of 10^{21} Pa s and a moderate viscosity increase in the lower mantle by a factor of 2. However, these previous results were derived from simplified four-layer or five-layer earth models, where the density contrasts at the internal boundaries were fixed according to PREM instead of the layers having volumetric density averages derived from PREM. Our findings are in complete agreement with a similar analysis of relative sea-level data by Mitrovica (1996), who, with a more rigorous scheme based on the resolving power of relative sea-level kernels, has demonstrated that the value of 10^{21} Pa s is appropriate for a region of the mantle extending to greater depths than the upper mantle defined by seismology. The value of 10^{21} Pa s, first proposed by Haskell (1935), was in fact somehow arbitrarily limited to the upper mantle after modern seismology had located the boundary between the upper and lower mantles at 670 km, as pointed out by Mitrovica (1996).

4.2 UPPER MANTLE VISCOSITIES LOWER THAN 10^{21} PA S

In the previous section we saw that within the framework of a multilayered, viscoelastic earth model based on PREM a simultaneous fit to TPW and $\dot{J}_2$ data is not possible when the viscosity value of 10^{21} Pa s is limited to the upper mantle. In this section the depth of the viscosity contrast is fixed at 670 km and the upper mantle viscosity is modified in order to get a simultaneous fit with TPW and $\dot{J}_2$ data when the lower mantle viscosity is varied from 10^{21} Pa s to 10^{24} Pa s, as in the previous figures.

In Figure 4.9 the upper mantle viscosity is reduced from 10^{21} Pa s (dotted curve) to 10^{20} Pa s (solid curve); intermediate values are 5×10^{20} Pa s (short-dashed curve) and 2×10^{20} Pa s (long-dashed curve). The reduction in the upper mantle viscosity diminishes the TPW signal in the whole range of viscosities, but it should be noted that this occurs in a strongly nonlinear fashion. A reduction factor of 2 from 10^{21} Pa s to 5×10^{20} Pa s has minor effects, except for a small increase in the proximity of the local maximum at 10^{22} Pa s. A decrease in the upper mantle viscosity from 5×10^{20} Pa s to 2×10^{20} Pa s causes a further increase in proximity of the local maximum and a decrease in the range of lower mantle viscosities from 10^{21} Pa s to 4×10^{21} Pa s. This reduction in the signal is confirmed by the upper mantle viscosity of 10^{20} Pa s, which provides the smallest TPW values for lower mantle viscosities smaller than 10^{22} Pa s. Although in a nonlinear fashion, the reduction of the upper mantle viscosity has the effect of displacing the TPW curve to the left. Except for the lowest upper mantle viscosity of 10^{20} Pa s, which predicts a lower mantle of 2.5×10^{21} Pa s, the inference of the viscosity value of the lower mantle is not affected by a

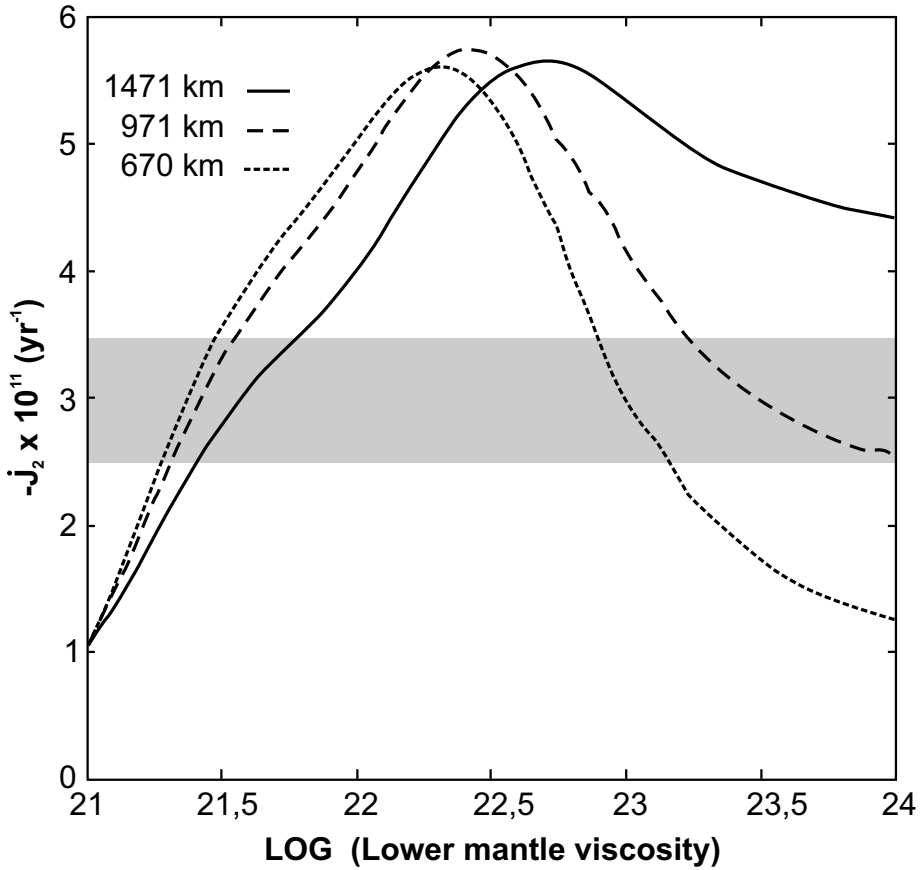


Figure 4.8. Effects of the depth of the viscosity contrast on the present-day secular variation in $\dot{J}_2$ as a function of the value of the viscosity of the lower mantle. The standard 31-layer model in Table 4.1 is used as the earth model. The three cases depict the results for the boundary between the upper and lower mantles (upper and lower mantles are defined with respect to viscosity) at 1471, 971, and 670 km depth. The upper mantle has a viscosity of 10^{21} Pa s. The hatching has the same meaning as in Figure 4.6.

reduction factor of 5 from the reference value of 10^{21} Pa s. The reduction of the upper mantle viscosity has, on the other hand, the major effect of increasing the viscosity contrast at 670 km from 3 for the short-dashed curve, corresponding to 10^{21} Pa s, to 8 or 20 for the upper mantle viscosities of 5×10^{20} and 2×10^{20} Pa s, respectively.

In Figure 4.10, the same analysis is carried out for $\dot{J}_2$. The curves are the same as in Figure 4.9. As in the case of Figure 4.8, the major effects of the viscosity decrease are visible in the high-value region of the lower mantle viscosities for

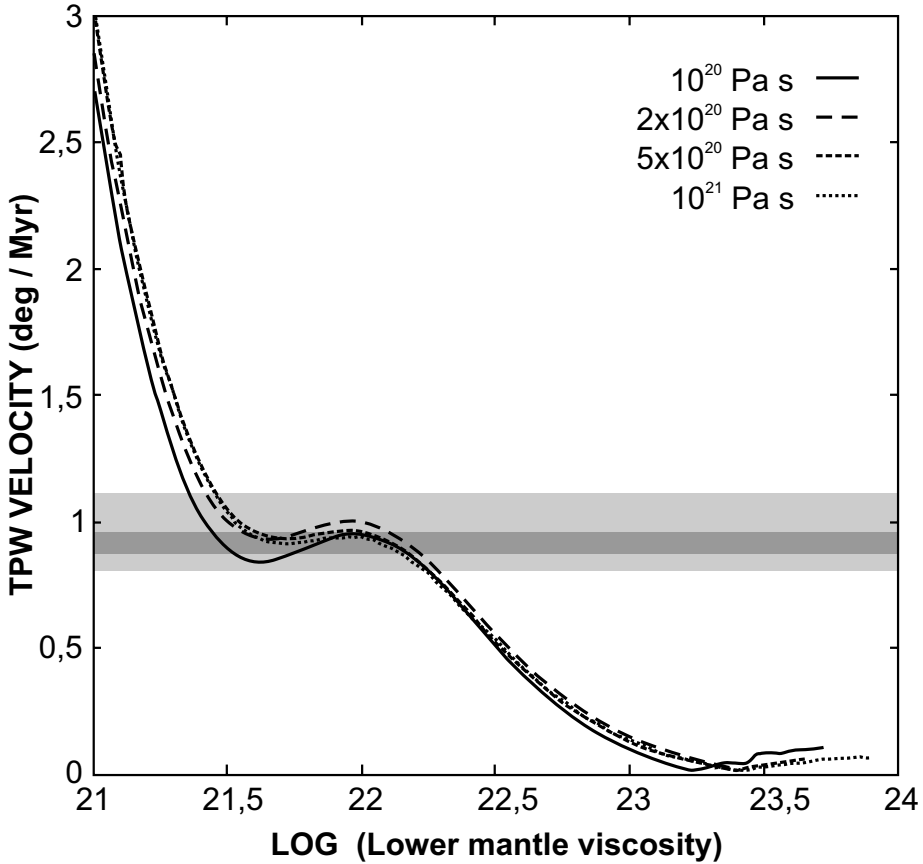


Figure 4.9. Effects of a lower value for the upper mantle viscosity on present-day true polar wander as a function of the value of the viscosity of the lower mantle. The standard 31-layer model in Table 4.1 is used as the earth model. The solid curve depicts the case of an upper mantle viscosity of 10^{20} Pa s. The dotted curve depicts the case of an upper mantle viscosity of 10^{21} Pa s. The two cases for the upper mantle viscosity between these two values are depicted by the short-dashed curve for the case of 5×10^{20} Pa s and by the long-dashed curve for the case of 2×10^{20} Pa s. The hatching has the same meaning as in Figure 4.5.

viscosities higher than 10^{23} Pa s. Major modifications with respect to the upper mantle viscosity of 10^{21} Pa s also occur in proximity to the maximum of the curves and for lower mantle viscosities higher than 3×10^{21} Pa s.

The reduction of the peak value located at the lower mantle viscosity of 3×10^{22} Pa s and the increase in the signal for lower mantle viscosities larger than 10^{23} Pa s agree with the results obtained by Mitrovica and Peltier (1993). The reduction in the upper mantle viscosity is responsible for an increase in the range of admissible lower mantle viscosities from $1.5 - 3 \times 10^{21}$ to $1.5 - 4 \times 10^{21}$

Pa s for an upper mantle viscosity of 10^{20} Pa s. With respect to the upper mantle value of 10^{21} Pa s, the $\dot{J}_2$ curves for reduced upper mantle viscosities are displaced to the right. This allows a simultaneous fit with the TPW data of McCarthy and Luzum (1996) for a viscosity increase at 670 km, which can be as high as factors of 20 and 30 for upper mantle viscosities of 2×10^{20} and 10^{20} Pa s, respectively. It should be noted that a simultaneous fit of TPW and $\dot{J}_2$ for an upper mantle viscosity of 5×10^{20} Pa s can only be obtained with the data of Dickman (1977), which have larger error bounds.

In comparison with the recent TPW data by McCarthy and Luzum (1996), those by Dickman (1977) are less discriminating in ruling out a viscosity value of 10^{21} Pa s above the 670 km discontinuity, although a clear tendency to prefer a lower value for the upper mantle viscosity is also shown by Dickman's (1977) data by means of enlarging the interval of admissible viscosity solutions for the two data sets.

The upper and lower mantle viscosities of $1 - 5 \times 10^{20}$ and $2.5 - 4 \times 10^{21}$ Pa s inferred from the McCarthy and Luzum (1996) TPW data are consistent with the estimates made by Lambeck *et al.* (1990) of $3 - 5 \times 10^{20}$ and $2 - 7 \times 10^{21}$ Pa s for these parameters, although our results have the tendency to remain in the lower limit of Lambeck *et al.*'s (1990) predictions. This tendency is probably attributable to model differences with Lambeck *et al.*'s (1990) analysis, where attention is focused on a different postglacial rebound signal, such as sea level in the far field, and compressible viscoelastic models with a lithospheric thickness smaller than ours. If we focus on the viscosity contrast at the boundary between the upper and lower mantles, we obtain a complete agreement with the findings of Lambeck *et al.* (1990), whose preferred upper and lower mantle viscosity contrast ranges between a factor of 4 and 25.

The results shown in this section indicate that the different viscosity models obtained in the past are mainly a consequence of a different way of mantle stratification. For an upper mantle viscosity varying between 10^{20} Pa s and 10^{21} Pa s, the shape of the TPW and $\dot{J}_2$ curves as a function of the viscosity of the lower mantle agrees with both previous analytical results based on simplified mantle models (Yuen and Sabadini, 1984; Spada *et al.*, 1992b) and recent findings by Milne and Mitrovica (1996). The TPW curves show a non-monotonic decrease in the signal from about 3° Myr^{-1} for an isoviscous mantle to zero for lower mantle viscosities higher than 10^{23} Pa s, with a characteristic upwarping in proximity of 10^{22} Pa s. The local maximum in TPW simulations is responsible for a multiplicity of solutions in the inference of lower mantle viscosity from 3×10^{21} to 10^{22} Pa s. These findings are in distinct contrast with recent analyses by Peltier and Jiang (1996), who, for viscosity contrasts located at 670 km, predict that TPW is negligible in the proximity of 10^{22} Pa s, where our local maximum of $0.9^\circ \text{ Myr}^{-1}$ is located. When the depth of the viscosity contrast is increased from 670 km to 1471 km in the lower mantle, the multiplicity of

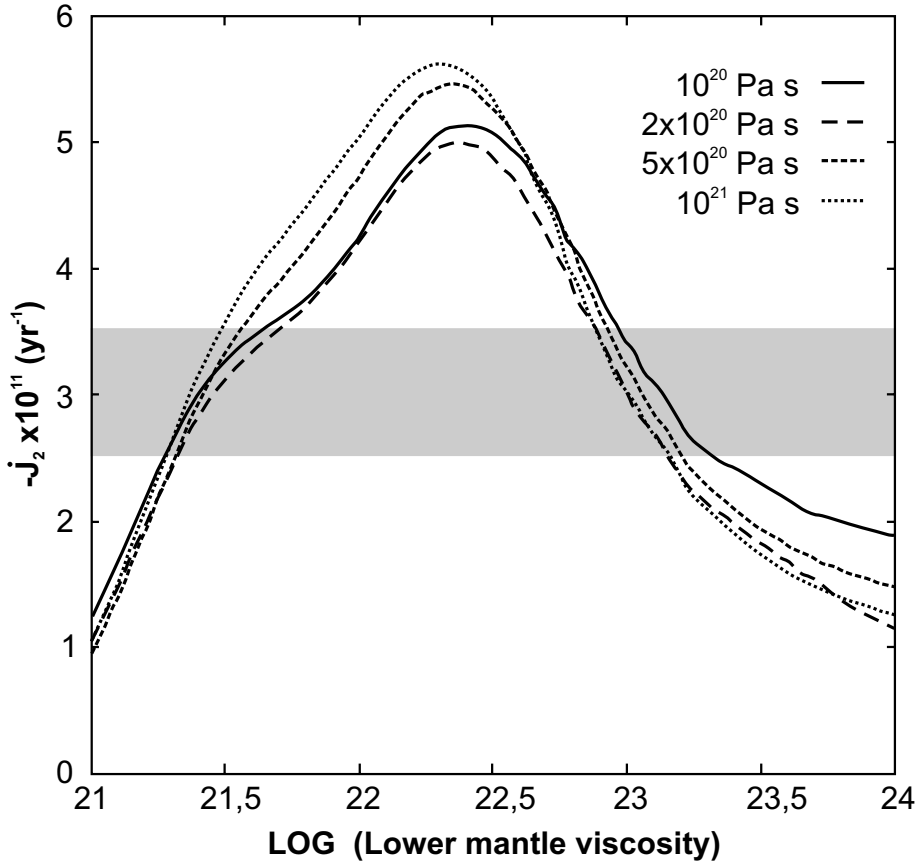


Figure 4.10. Effects of a lower value for upper mantle viscosity on present-day secular variation in $\dot{J}_2$ as a function of the value of the viscosity of the lower mantle. The standard 31-layer model in Table 4.1 is used as the earth model. The solid curve depicts the case of an upper mantle viscosity of 10^{20} Pa s. The dotted curve depicts the case of an upper mantle viscosity of 10^{21} Pa s. The two cases for the upper mantle viscosity in between these two values are depicted by the short-dashed curve for the case of 5×10^{20} Pa s and by the long-dashed curve for the case of 2×10^{20} Pa s. The hatching has the same meaning as in Figure 4.6.

solutions for both TPW and $\dot{J}_2$ in the inference of the viscosity of the lowermost portion of the mantle disappears. In the TPW curve this occurs via a reduction in the local maximum at 10^{22} Pa s at a single crossing of the observational data, while in $\dot{J}_2$ the high-branch viscosity solution disappears because of an increase in the signal beyond -3×10^{-11} yr $^{-1}$ for viscosities in the lowermost mantle higher than 3×10^{21} Pa s. If the viscosity contrast is located at depths of about 1400 km, TPW and $\dot{J}_2$ data allow for an unambiguous viscosity solu-

Table 4.2. Parameters for the 4-layer earth model in Yuen *et al.* (1983). r is the distance with respect to the center of the Earth, ρ the density of the layer, and μ the rigidity.

layer	r (km)	ρ (kg/m ³)	μ (N/m ²)	
1	6371 – 6271	2771	3.15×10^{10}	lithosphere
2	6271 – 5701	4120	9.54×10^{10}	upper mantle
3	5701 – 3480	4508	1.99×10^{11}	lower mantle
4	3480 – 0	10925	0	inviscid fluid core

tion for the upper portion of the mantle of 10^{21} Pa s, agreeing with Haskell's (1935) viscosity. A viscosity of 10^{21} Pa s limited to the upper mantle has been retrieved in some previous TPW and $\dot{J}_2$ analyses because of the underestimate of mantle buoyancy in simplified five-layer fixed boundary contrast models and of the trade-off between the density and the viscosity profile (e.g., Spada *et al.*, 1992b). If the viscosity contrast is located at 670 km depth, which defines the boundary between the upper and the lower mantles based on seismological data, TPW and $\dot{J}_2$ indicate a clear trend toward viscosities in the upper mantle lower than the Haskell value of 10^{21} Pa s. This increases the viscosity contrast to a factor of 20, in agreement with the far-field RSL analysis by Lambeck *et al.* (1990) and with the studies on the long-wavelength geoid anomalies supported by mantle convection (Richards and Hager, 1984). When analytical stratified models carrying all the buoyancy of PREM are considered, it becomes possible to explain some apparently contradictory results of recent inferences of mantle viscosity.

5. INFERENCE OF LITHOSPHERIC THICKNESS FROM TPW AND LENGTH OF DAY VARIATIONS

The modeled rotational response of the Earth to Pleistocene deglaciation can be used to constrain not only the viscosity of the mantle, as done in the previous section 4.4, but also the thickness of the lithosphere. This attempt was first carried out by Yuen *et al.* (1983), where the 4-layer model in Table 4.2 was used, rather than the more sophisticated multilayer models presented in this book, to infer the global mechanical properties of the lithosphere by a simultaneous fit of TPW data of the International Latitude Service (Dickmann, 1977) and length of day (l.o.d.) data by Muller and Stephenson (1975) instead of the $\dot{J}_2$ obtained from geodetic satellites. The results are shown in Figure 4.11, where the permissible lower mantle viscosity solutions and lithospheric thickness are shown simultaneously for the I.L.S. and l.o.d. data, dark grey and dotted patterns, respectively.

The region of the parameter space delineating the lithospheric thickness, in which a viscosity solution for the lower mantle exists, is given by the region

where I.L.S. and l.o.d. data are fitted simultaneously, where the dark grey and dotted patterns are superimposed, with l.b. and u.b. denoting the lower and upper bounds of the solutions that fit the lower and upper error bounds of the observational data. The load model used in these calculations is given in Yuen *et al.* (1982), with a total mass of the Laurentian and Fennoscandian ice sheets discharged in the oceans equivalent to a sea-level rise of 90 m, as indicated by the parameter p_1 . From the top to the bottom panel the time t_1 elapsed after the end of deglaciation is varied from 4.5×10^3 yr to 7×10^3 yr. Inspection of this figure shows that l.o.d. solutions reveal very little sensitivity to lithospheric thickness variations. I.L.S. data, on the contrary, show a very high sensitivity to both viscosity and lithospheric thickness variations, as already anticipated in Chapter 3 and Section 4.1, due to the concomitant effects of centrifugal and isostatic relaxation.

This basic difference in the sensitivity of the two solutions to changes in input parameters of the system can thus be traced to the inherently more complicated structure of the solutions to the polar wander equations (3.57). These l.o.d. solutions can on the other hand be used as constraints on the set of valid I.L.S. solutions, which for this purposes have been restricted to those which can match the two extreme values of the I.L.S. data (Dickman, 1977). Using this as a criterion, it is possible to regard the intersection of the two families of solutions in Figure 4.11 as valid in the viscosity and lithospheric thickness space. For $t_1 = 6 \times 10^3$ yr, the lower mantle viscosity ν_2 varies between 10^{21} Pa s and 2.5×10^{21} Pa s and the admissible solutions of the lithospheric thickness lie in the range between 130 and 200 km. For $t_1 = 4.5 \times 10^3$ yr, the lower mantle viscosity is closer to 10^{21} Pa s and the allowable lithospheric thickness smaller, ranging between 100 and 170 km. This set of values represented the first attempt, albeit crude, ever made to estimate the thickness of the lithosphere using the rotational data set alone (Yuen *et al.*, 1983). These findings on the global averaged lithosphere have important implications, leading to the suggestion that strong lateral heterogeneities exist beneath oceanic and continental regions in the upper mantle. Indeed, if we accept other geophysical evidence, such as heat flow and sea floor topography (e.g., Parsons and Sclater, 1977), that the averaged thickness of the oceanic lithosphere is about 100 km, then the above inferences on the global lithospheric thickness would favor a continental plate at least twice as thick as its oceanic counterpart.

6. ICE AGE CYCLES AND THE POLAR WANDER PATH

In the former section present-day polar wander as a consequence of post-glacial rebound was considered. Figure 4.12 shows instead the modeled polar wander over the complete ice-age cycles period in Figure 4.2, with the ice load of this figure as input. The case of an earth model with the top 120 km

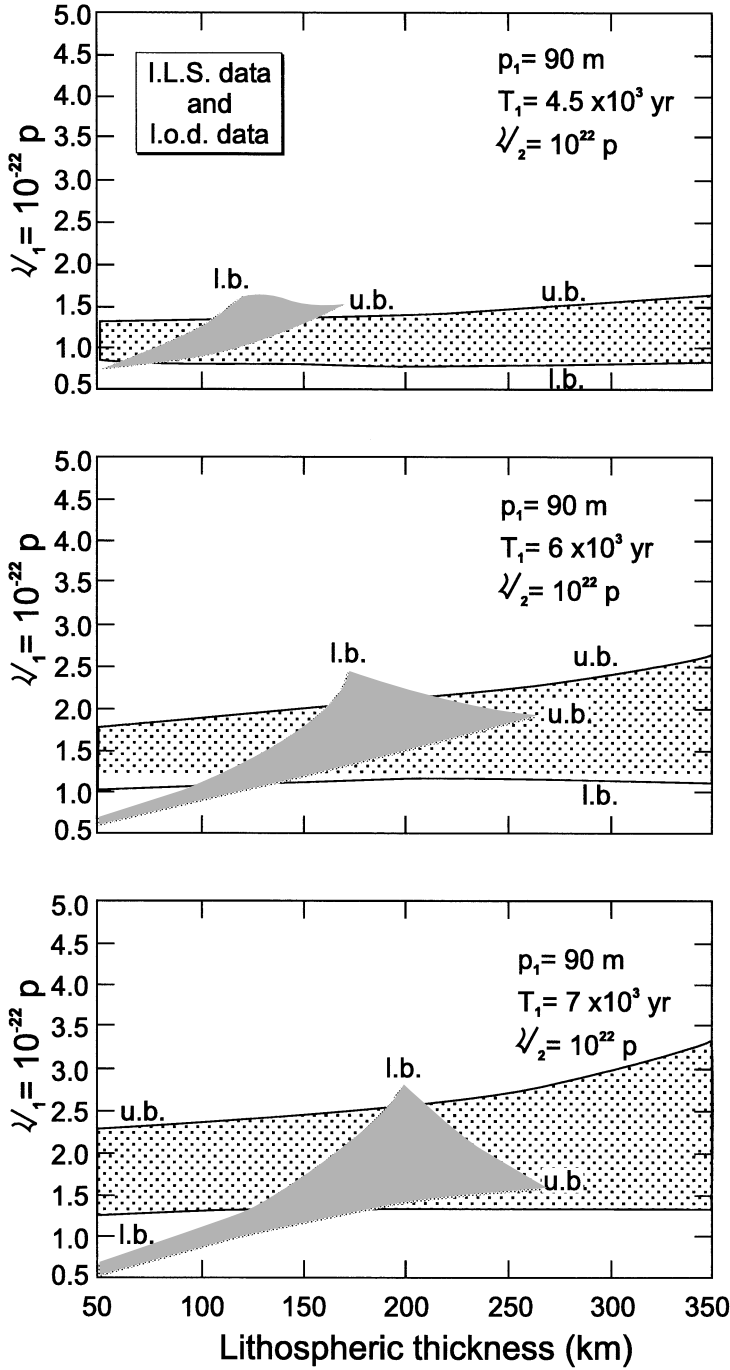


Figure 4.11. Permissible lower mantle viscosity solution as a function of lithospheric thickness. The viscosity of the upper mantle is fixed at 10^{21} Pa s . The symbols l.b. and u.b. denote the solutions that fit the lower and upper bounds of the rotational data. Hatched and circled patterns represent solutions obtained by fitting the calculations to I.L.S. and l.o.d. data (Dickman, 1977; Muller and Stephenson, 1977).

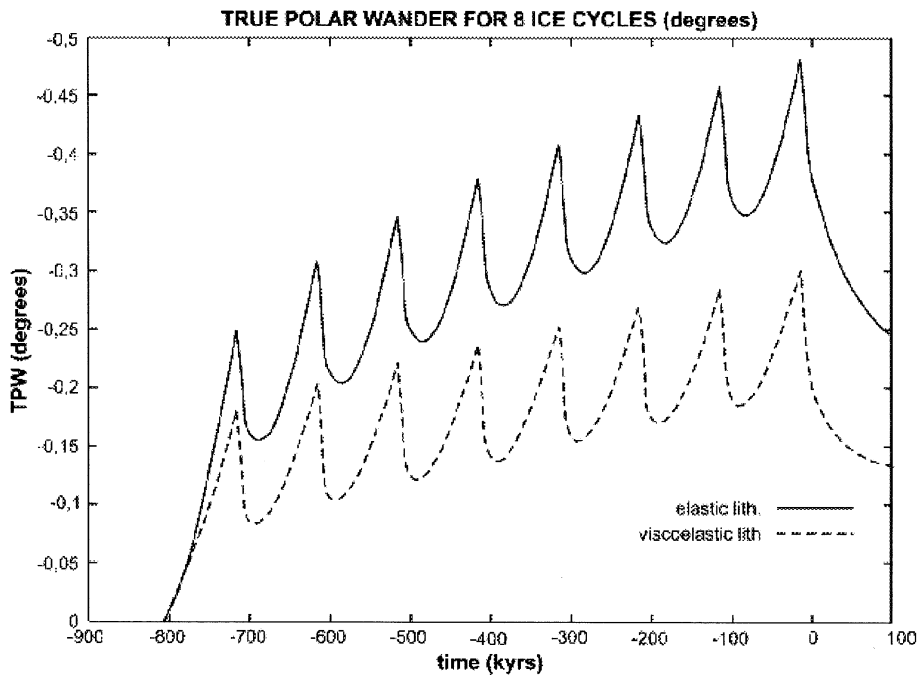


Figure 4.12. TPW as a function of time (the present is time 0) for the ice load forcing of Figure 4.2. The mantle viscosity is fixed at 10^{21} Pa s. The thickness of the lithosphere is 120 km. Two cases are depicted, for an elastic lithosphere, solid curve, and a viscoelastic one, dashed curve, with the same viscosity of the mantle. The 5-layer model in Table 2.2 has been used.

having both an elastic and viscoelastic rheology is modeled, solid and dashed curves, respectively. The viscosity of the mantle (and lithosphere in the case of the viscoelastic top layer) is 10^{21} Pa s. The number of layers in the solid-earth model is five (volume-averaged parameters for homogeneous lithosphere, shallow upper mantle, transition zone, lower mantle and core, Table 2.2). Figure 4.12 shows several interesting aspects. It is clear that the question whether the lithosphere is purely elastic or whether it has a finite viscosity is an important one: the long-term trend differs by about a factor of two. Another point that immediately strikes the eye is that there is a short-term trend (the saw-tooth profiles on time scales of about 100,000 years) and a long-term trend. The long-term trend is not zero, the average position of the rotation axis gradually shifting further away from its initial position 806,000 years ago as time proceeds up to the present. One might perhaps think that this is due to the fact that the waxing phases of the ice cycles are longer than the waning phases.

Figure 4.13 shows that this is not the case. In Figure 4.13 the time of a phase of ice growth has been taken as being equal to the time of a phase of ice

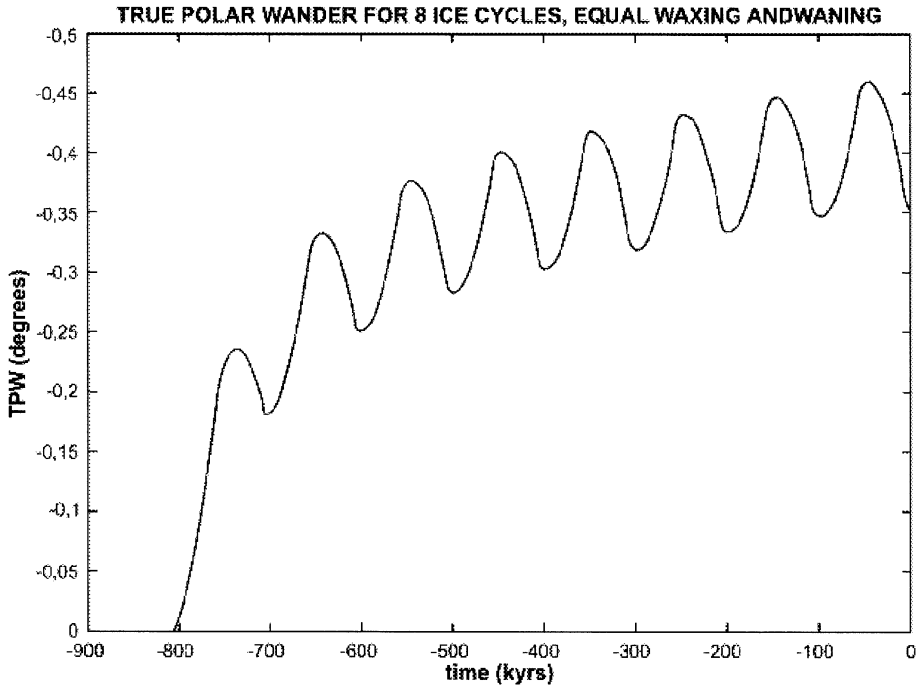


Figure 4.13. TPW as a function of time for the ice load forcing history in Figure 4.2 and an elastic lithosphere, but now with equal waxing and waning time scales of 50,000 years. The mantle viscosity is uniformly 10^{21} Pa s.

decay: both have a period of 50,000 years. A comparison of Figure 4.12 with Figure 4.13 (the case of the lithosphere being elastic is modeled) shows that the average polar wander (the long-term trend) is the same; only the short-term trend shows differences in the sense that the 'saw-teeth' of the corresponding curve in Figure 4.12 have been smoothed. The reason why there is a net, almost linearly increasing, long-term TPW trend is that after an ice cycle has been completed the Earth is still out of isostatic equilibrium. The relaxation times of some of the modes by which the Earth relaxes to the changed surface mass distribution are even longer than the periods of the ice cycles. Thus it also becomes understandable why in the case of an elastic lithosphere in Figure 4.12 the displacement of the polar axis is greater than in the case of a viscoelastic lithosphere: an elastic lithosphere prohibits complete relaxation even on long time scales, whereas a viscoelastic lithosphere will relax completely inside a finite time.

In Figures 4.12 and 4.13 only the magnitude of the polar wander triggered by the Pleistocene Ice Ages has been plotted, not the direction. Immediately

after the ice begins to grow 806,000 years before the present, the direction of polar wander is toward Russia. This is understandable, as the ice load that has most effect on the rotation axis is the one over Canada, the Laurentide. When this ice sheet begins to form, there will thus be an excess mass over Canada. A deformable rotating body will react to an excess mass through centrifugal force by moving its rotation axis in such a way that the excess mass will be as far away from the rotation axis as possible. Thus the rotation axis moves in the direction opposite to where the surface load is until the maximum of the ice load has been reached. When the ice starts to melt again, it is a question of how much the ice load has been compensated by solid-Earth deformation: if the Earth relaxes fast to the ice load (deviations from isostatic equilibrium remaining small during ice build-up), then the situation during ice melt will readily be such that ice melt will overcompensate the positive anomaly that was formed during times of ice growth. In the situation in Figure 4.12 this overcompensation for the positive anomaly proceeds indeed so fast that it gives rise to the sharp saw-teeth during times of ice maximum. During times of ice minimum, the overcompensation (from a negative to a positive anomaly) proceeds somewhat slowly, giving rise to the smooth reversals at the bottom curves in Figure 4.12. In Figure 4.13, with the equal waxing and waning periods, the relaxation of the solid Earth is such that the overcompensation at times of ice maximum takes longer so that the sharp teeth in Figure 4.12 have been replaced by smooth curvings. The direction of polar wander during ice melt is about 180 degrees opposite to the direction during ice growth, that is, during periods of ice melt the rotation axis moves in the direction of Canada (as it is doing at present). The long-term TPW direction is, according to figures 4.12 and 4.13, in the same direction as it is during periods of ice growth, so in the direction of Russia.

Whether polar wander is mainly caused by ice age cycles, by mantle convection or tectonics is thus an important question when considering what is to be studied: if the present-day (secular) polar wander is to be studied, then the direction of polar wander induced by the Pleistocene ice age cycles is toward Canada; if the polar wander path over the whole Pleistocene is to be considered, then the direction is toward Russia. Note that the magnitudes of the long-term trend are an order of magnitude smaller than the magnitudes during an ice cycle. This implies that if both mantle convection and ice mass variations were to have a comparable influence on driving polar wander during ice ages on a time scale of one million years, on time scales of 10,000 - 100,000 years ice mass variations would have a far greater influence than mantle convection. The question of whether mantle convection or ice age cycles is the main driver of polar wander during recent times is as old as the first observations of the secular drift of the rotation axis. Apart from the direction discussed above, Figure 4.12 might offer another observable that could distinguish between ice or mantle convection being responsible for TPW. If polar wander were triggered by sub-

duction of ocean lithosphere, the polar wander trend would be a smooth one over millions of years (e.g., Spada *et al.*, 1992a), without the saw-teeth on a 100,000 year time scale that are triggered by ice ages. Might such saw-teeth be discerned from paleomagnetic data? Although it would be an extremely important observation, chances that we might ever detect them are very slim. If we consider the error bars in the TPW data of Besse and Courtillot (1991) and compare them with the 0.1 degree accuracy in Figure 4.12 that is needed to discern the to-and-fro movements of the rotation axis, then we learn that there is a discrepancy in accuracy of one to two decades. A lowering of the standard deviations by one to two orders of magnitude seems to be virtually impossible. A polar wander of 0.1 degree might even be below the threshold of intrinsic errors of paleomagnetic field measurements (C. Duermeijer, personal communication), so it is very unlikely that such small magnitudes might ever be discernible from paleomagnetic observations. One last aspect in Figure 4.12 and 4.13 worth mentioning concerns the behavior of polar wander in the future. It is clear that if a new ice age period were to begin, that is, if a 'new Laurentide ice sheet' were to form over Canada, then the long-term trend before the present would resume after time 0 in Figure 4.12. However, if there were not to be another ice age period, then the polar wander would be as depicted in Figure 4.12 for times after time 0. The movement of the rotation axis would be such that it would relax towards the original position it had left 806,000 years before the present (R. Gordon, personal communication). Although it is not visible in the figure, there still would be a difference between the two cases depicted in Figure 4.12 after a long time had elapsed: in the case of an earth model with an elastic lithosphere, the rotation axis would go on moving in the direction it is moving at present, that is, it would move on until the negative gravity anomaly created by the vanished ice sheet over Canada were on the North Pole. In the case of an earth model with a viscoelastic lithosphere, the movement of the rotation axis would stop if isostasy were restored. Here we also see how important the question of the rheology of the lithosphere is for the movement of the rotation axis. The curves in Figures 4.12 and 4.13 are based on a 5-layer model and a mantle with a uniform viscosity of 10^{21} Pa s. In the previous section 4.4 we saw that the viscosity of the upper mantle is more likely to be lower than this value, and the viscosity of the lower higher. The effects of both layering and viscosity can be seen in Figure 4.14.

Here we have used the 31-layer solid-earth model in Table 4.1. The lithosphere is elastic. The solid curve depicts the case of a uniform viscosity of 10^{21} Pa s again. Although at first sight the differences with the corresponding curve in Figure 4.12, produced with a 5-layer earth model, might not seem to be dramatic, the long-term TPW rate does differ considerably between the two uniform 10^{21} Pa s cases upon closer inspection (to see this the reader might put a ruler in such a way as to connect the peaks in each of the curves). More

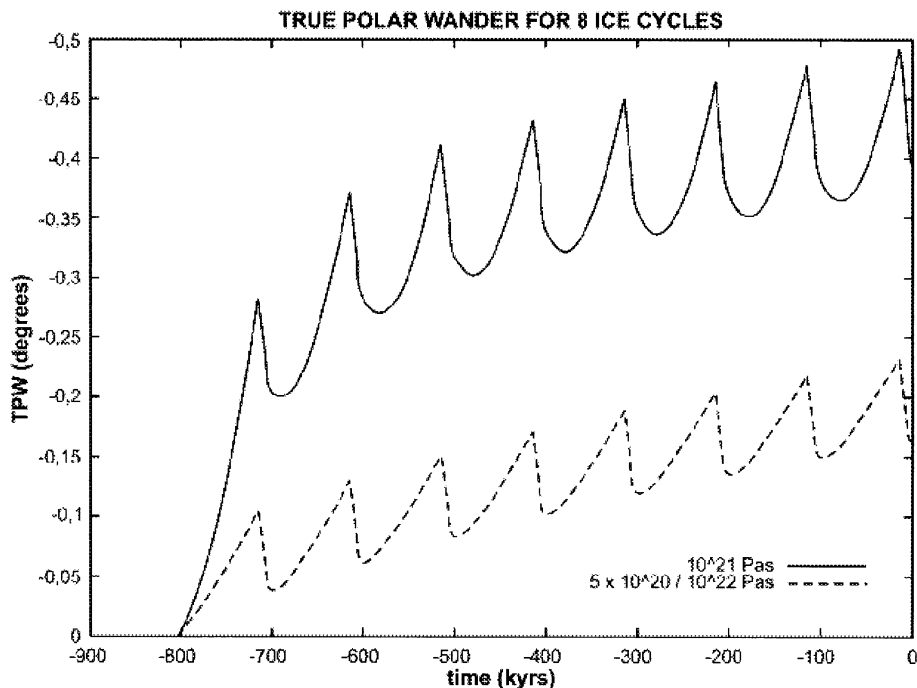


Figure 4.14. TPW as a function of time (the present is time 0) for the ice load forcing of Figure 4.2 and 31-layer earth model in Table 4.1. Two cases are depicted: the solid curve for a homogeneous mantle viscosity of 10^{21} Pa s; the dashed curve for the case of an upper mantle viscosity of 5×10^{20} Pa s and a lower mantle viscosity of 10^{22} Pa s.

eye-striking in Figure 4.14 is the huge reduction of polar wander for the case of a lower mantle viscosity of 10^{22} Pa s.

7. INFLUENCES OF PHASE-CHANGE VS. CHEMICAL STRATIFICATION

In the previous sections we considered the effects of viscosity and density stratification. We now deal with the effects of chemical and phase-change density stratification in the mantle on present-day TPW, on the basis of analytical, multilayered earth models described previously.

Some cases are devoted to the study of two 31-layer end-member models, with the upper and lower mantle away from the transition zone stratified in terms of chemical (fully non-adiabatic) density contrasts or in terms of phase-change density contrasts, fully adiabatic. Both models evolve from an initial state of self-compression, as shown by the thick dashed curve in Figure 4.15, but the

deformation induced by ice load is fully non-adiabatic or adiabatic for the two cases.

The 31-layer fully adiabatic ($\xi = 1$ in Section 1.7) phase-change model shows the effects of a fully adiabatic mantle with respect to the chemically stratified mantle in the previous sections of this chapter. Since adiabatic phase changes in the portions of the mantle away from the transition zones cannot be excluded, this model helps to understand the effects of these possible phase changes on TPW and, in particular, to show how TPW simulations depend on the nature of the density contrasts within the mantle.

As we did in Figure 1.6, we can now analyze the effects of phase changes on the rotational inverse relaxation times. Figure 4.16 portrays the $-1/a_i$ entering equation (3.61) as a function of the lower mantle viscosity, for a 5-layer chemically stratified mantle, top panel, and a phase change corresponding to $\xi = 0.7$, with density stratification provided in Table 2.2.

Comparison between Figures 1.6 and 4.16 indicates that the rotational inverse relaxation times are faster than the companion isostatic ones. Furthermore, comparison of the two panels in Figure 4.16 indicates that the effect of the phase change is to increase the relaxation time of the rotational modes.

Figure 4.17 involves a comparison between the 31- and 5-layer chemically stratified mantle (solid and dashed curves, respectively) and the 31- and 5-layer phase change mantle (dotted and dash-dotted curves). These models are described in Tables 4.1 and 2.2, where the density has been volume-averaged from the PREM. The density contrasts is treated as shown by equations (1.106) and (1.112), for chemical stratification and phase changes, respectively. Although the two models have a different number of layers, they are both based on PREM and they are thus characterized by a large amount of density contrast and by the same gravity at the surface.

Deviations between the two chemically stratified models, solid and dashed curves, occur for lower mantle viscosity higher than 21.5 in logarithmic scale, and in particular over the almost flat region between 21.5 and 22.0 that characterizes the TPW curves for present day polar wander, as discussed at length in section 4.4. The agreement between the two models tells us that as far as Earth rotation is concerned, it does not matter where the chemical density contrasts are located within the mantle, once the density is volume averaged from PREM and thus the gravity is correctly reproduced. This result should not be surprising, since a global dynamic process, such as Earth rotation, is not sensitive to the detailed distribution with depth of the density discontinuity.

The 31- and 5-layer models maintain this similarity even in the case in which all the interfaces in Tables 4.1 and 2.2 behave like a phase-change boundary with the same response factor $\xi = 0.7$, as portrayed by the dotted and dash-dotted curves, corresponding to the 31-layer and the 5-layer models. The peak values of TPW, obtained for $\nu_2 = 10^{21}$ Pa s, are reduced by a factor two for

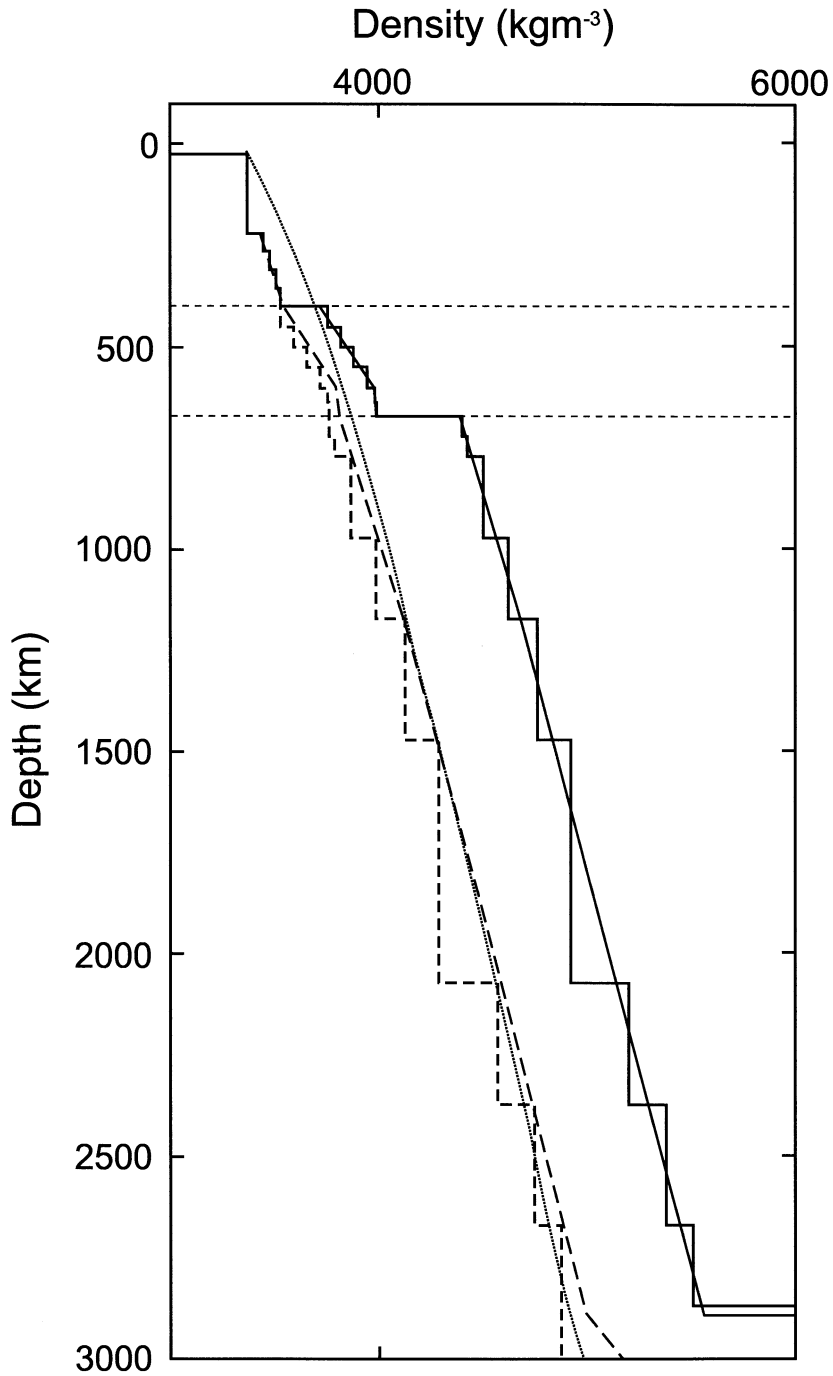


Figure 4.15. Density profile of the PREM reference model, thin solid curve, and our 31-layer model, thick solid curve. In the dashed curves the density discontinuities at 420 and 670 km have been deleted in order to facilitate the comparison between the PREM and our 31-layer model with the density profile corresponding to the adiabatic density profile, thin dotted curve. Redrawn from Figure 1 in Sabadini *et al.* (2002).

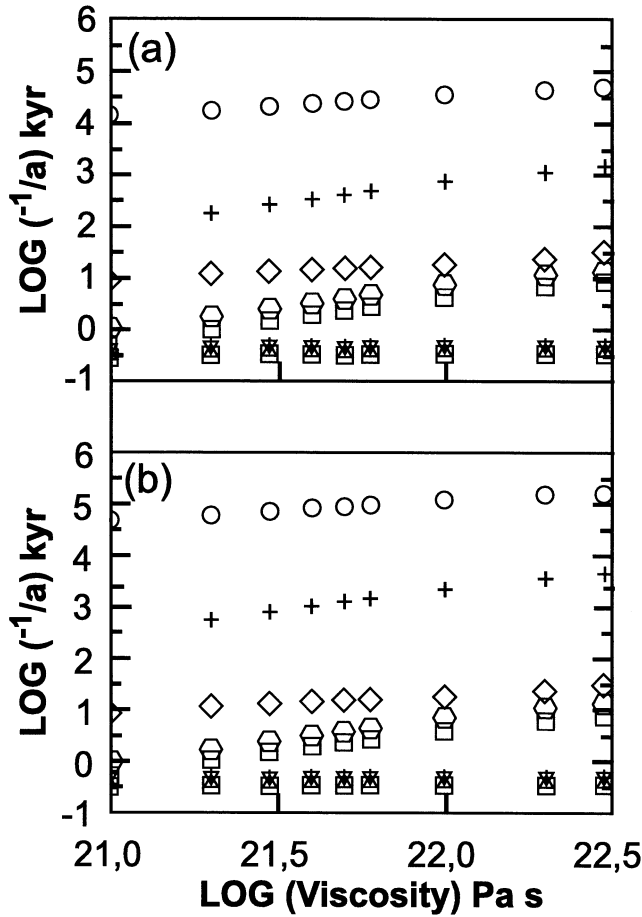


Figure 4.16. Inverse relaxation times for the rotational modes, equation (3.54), for the 5-layer model in Table 2.2, for chemical and phase-change boundary conditions at 420 and 670 km, panels (a) and (b), respectively. The phase change is characterized by $\xi = 0.7$. The circles and the crosses correspond to the 420 and 670 km discontinuities. Redrawn from Figure 3 in Sabadini *et al.* (2002).

the phase-change case with respect to the chemically stratified mantle. This reduction causes a substantial modification in the pattern of the TPW curves shown in section 4.4.

The TPW reduction for phase changes at 420 and 670 km with respect to the chemical boundaries obtained in Figure 4.17 is consistent with the results of Figure B1 in Johnston and Lambeck (1999). For viscosities close to the second relative maximum of 10^{22} Pa s, the reduction of the phase-change model with respect to the chemically stratified ones is about 30 per cent.

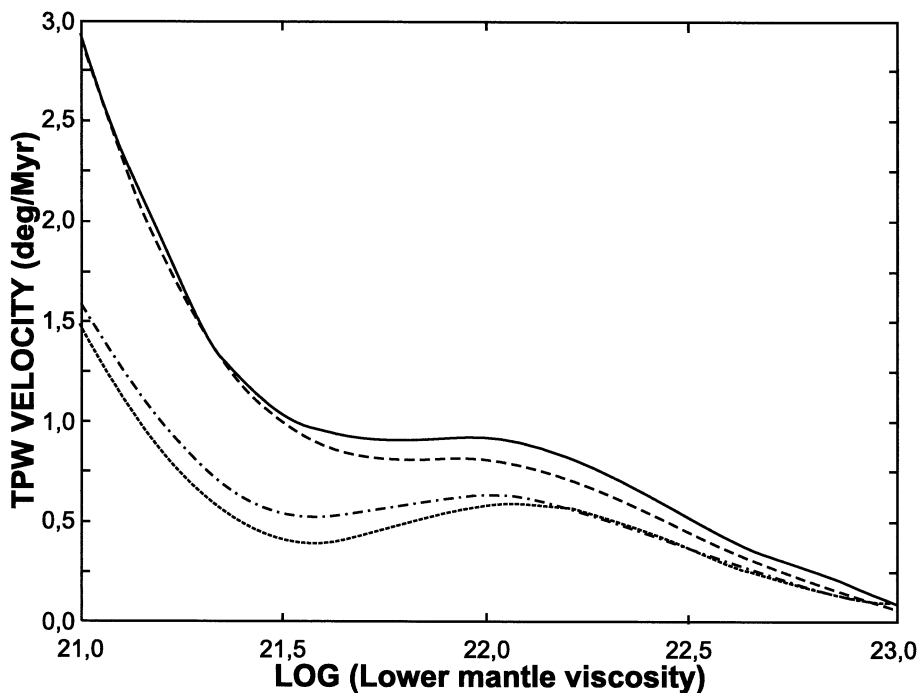


Figure 4.17. Present-day TPW as a function of lower mantle viscosity for the 31-layer earth model, solid and dotted curves (Table 4.1) and 5-layer model, dashed and dash-dotted curves (Table 2.2). The densities have been volume averaged from the PREM. The solid and dashed curves correspond to chemical density contrasts; phase-change density contrasts are portrayed by the dotted and dash-dotted curves, with $\xi = 0.7$. The upper mantle viscosity is fixed at 10^{21} Pa s. Redrawn from Figure 5 in Sabadini *et al.* (2002).

The deviations between the 31- and 5-layer phase-change models are visible only for lower mantle viscosities lower than 10^{22} Pa s; this indicates that in the lower mantle viscosity range $10^{21} \leq \nu_2 \leq 10^{22}$ Pa s, the rotational disequilibrium of the planet, responsible for TPW, is maintained by the density contrasts, while for $\nu_2 \geq 10^{22}$ Pa s the stiffening of the lower mantle overcomes the effects of the buoyancy forces at internal interfaces.

In Figure 4.18 the effects of the modification in the boundary conditions at the inner interfaces is considered in greater detail. The dotted curve is based on the 31-layer-model in which the density contrasts in the upper mantle between the base of the lithosphere and 420 km and in the whole lower mantle correspond to fully adiabatic phase changes, with $\xi = 1$ (see Figure 4.15). In the transition zone, including the discontinuities at 420 and 670 km, we assume that the mantle material is subject to phase changes, with $\xi = 0.7$. All the interfaces within the transition zone involve the density contrasts given in Table 4.1 and

we thus assume that each of them corresponds to a phase change. This choice is motivated by the fact that Figure 4.15 shows a density gradient in the transition zone higher than the adiabatic one due to self-compression. For simplicity, we assume the same response factor at each interface.

Comparison of this dotted curve with the results of the solid one representing a chemically stratified mantle, redrawn from Figure 4.17, confirms the substantial reduction in the TPW signal between these two end-member models, from a fully non-adiabatic mantle (chemically stratified) to a fully adiabatic mantle (phase change with $\xi = 1$). For viscosities of 10^{21} Pa s, the reduction is of a factor 4, while for higher viscosities of 10^{22} Pa s the reduction is limited to a factor 2. Comparison between these two models must be taken with caution, since they are both unlikely to be realistic for the mantle, although the phase-change one with $\xi = 1$ has the advantage of being fully adiabatic away from the transition zone, as required by the PREM model.

The dashed curve is redrawn from Figure 4.17 when at each interface of the 31-layer model, including the upper mantle, the transition zone and the lower mantle, the response factor is fixed at $\xi = 0.7$ (dash-dotted curve in Figure 4.17). Comparison between the dashed and dotted curves illustrates the effect of increasing the response factor from 0.7 to 1 in the uppermost mantle and in the lower mantle, which causes a 50 per cent reduction in TPW for viscosities of 10^{21} Pa s.

Comparison between the solid, dashed and dotted curves illustrates the large effect of a realistic adiabatic mantle in reducing the TPW signal of a chemical mantle.

Next, we return to a comparison of the 31-layer and the 5-layer model results, which requires that the uppermost mantle and the lower mantle are fully adiabatic and that both the density increase within the transition zone and the response factor due to the phase changes are the same for the two models. The only way to simulate the conditions of adiabaticity by means of a 5-layer model is to assume that the density in the uppermost mantle and in the lower mantle is constant; this automatically implies that one of the two conditions for a self-consistent comparison between the 31-layer and 5-layer models that we have considered above, namely that the gravity is the same for the two models, is now violated. Furthermore, the density contrasts in the 31-layer model away from the transition zone, although fully adiabatic, correspond to phase changes, which are not present in the 5-layer model. Since we have verified the important role played by the density contrasts within the mantle on TPW, we assume within the 5-layer model that the density contrast in the transition zone equals that of the 31-layer model, and that the density discontinuities at 420 and 670 km of the 5-layer model are chosen in such a way that the total density variation of the 5-layer model matches that of the 31-layer model, which is 868 kg/m^3 .

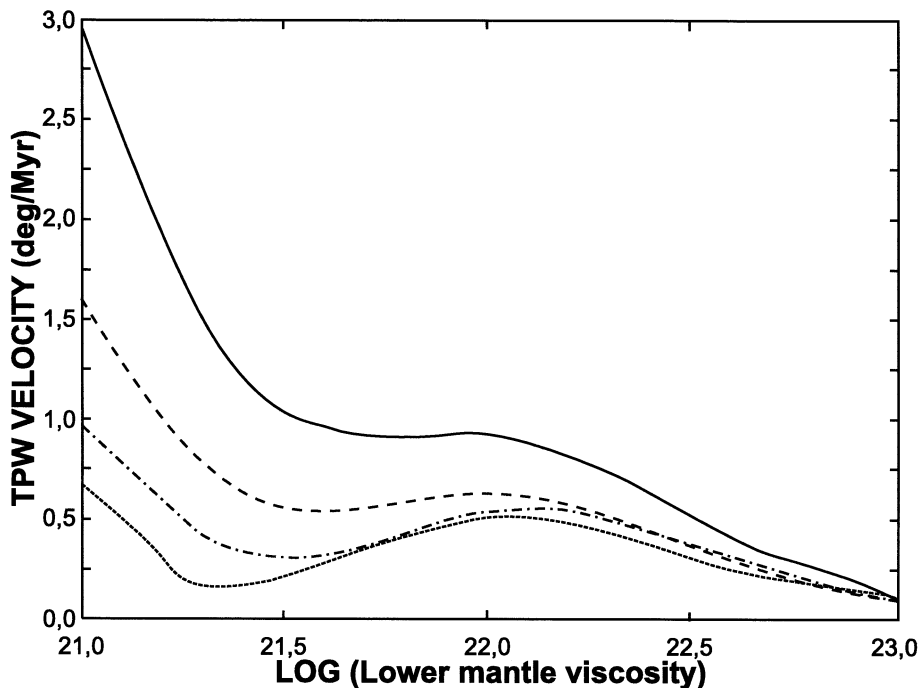


Figure 4.18. Present-day TPW as a function of lower mantle viscosity for the 31-layer earth model (Table 4.1), dotted, solid and dashed, and 5-layer model, dash-dotted, (Table 4.3). The solid curve is the same as in Figure 4.17, fully non-adiabatic, the dashed one is the same 31-layer model of Figure 4.17, with $\xi = 0.7$, while the dotted curve corresponds to $\xi = 1$ in the uppermost mantle and lower mantle, and $\xi = 0.7$ in the transition zone, as for the dash-dotted curve. The upper mantle viscosity is fixed at 10^{21} Pa s. Redrawn from Figure 6 in Sabadini *et al.* (2002).

Table 4.3. Parameters for the 5-layer fixed-boundary contrast Earth model. r is the distance with respect to the center of the Earth, ρ the density of the layer, and μ the rigidity.

layer	r (km)	ρ (kg/m ³)	μ (N/m ²)	
1	6371 – 6271	3184	7.28×10^{10}	lithosphere
2	6271 – 5951	3437	9.54×10^{10}	shallow upper mantle
3	5951 – 5701	3872	1.10×10^{11}	transition zone
4	5701 – 3480	4306	1.99×10^{11}	lower mantle
5	3480 – 0	10925	0	inviscid fluid core

The parameters of this 5-layer model are given in Table 4.3, to be compared with the 31-layer model in Table 4.1.

The results of this hybrid 5-layer model are given in Figure 4.18 by the dash-dotted curve. The similarity in the rotational behavior carried out by the 31-layer and 5-layer models, portrayed by the dotted and dash-dotted curves lying close to one another, suggests that the density contrasts in the upper and lower mantle away from the transition zone, once fully adiabatic, do not play a role in TPW. This result also shows that it is possible to simulate reasonably well the rotational behavior of a sophisticated, viscoelastic model carrying many layers by means of a simple 5-layer model, once the total density changes and response factor in the transition zone are the same for the two models. The 25 per cent difference between the 31- and 5-layer models for viscosities of the order of 10^{21} Pa s is attributable to their different gravity and to the nature of the density contrasts in the 31-layer model away from the transition zone, which, although fully adiabatic, are defined in terms of phase-change thermodynamics.

Since the 5-layer model does not carry any relaxation modes in the uppermost mantle or in the lower mantle, the similarity of the dotted and dash-dotted curves seems to indicate that the adiabatic behavior of a stratified mantle can be reproduced by deleting the buoyancy modes excited at the inner interfaces, as proposed by Ricard and Sabadini (1990) and more recently by Mitrovica and Milne (1998).

The validity of this hypothesis is verified in Figure 4.19 where the 5-layer model carrying the modes at 420 and 670 km corresponding to a response factor $\xi = 1$, solid curve, is compared with the same 5-layer model where the buoyancy modes arising from an assumption of chemical stratification at 420 and 670 km have been deleted, dashed curve. The two curves portray the same behavior for varying lower mantle viscosity, with at most a 20 per cent difference for viscosity of about 10^{22} Pa s. The shape of the TPW curves now resembles $\dot{J}_2$, the time derivative of the degree 2 component of the geopotential, as a function of the lower mantle viscosity, as shown for example in Figure 4.6. The dashed curve in our Figure 4.19 had previously been obtained by Mitrovica and Milne (1998), who were the first to show the effects of deleting the buoyancy mode associated with the (chemical) density contrast at 670 km on TPW simulation and to obtain the characteristic ‘inverted parabola’ in this figure.

The results in Figure 4.19 have a simple physical explanation: material crossing a fully adiabatic interface changes its density without experiencing any buoyant restoring force. The behavior of this fully adiabatic transition zone can thus be reproduced by simply deleting the buoyancy modes. Deviations between the two models occur in fact in the viscosity region 21.5–22.5, in logarithmic scale, where the stiffening of the lower mantle is the dominant mechanism for TPW excitation. The physics of a fully adiabatic interface can be well reproduced by simply deleting, in a chemically stratified mantle, the buoyancy modes arising from non-adiabatic boundary conditions, in agreement with the concept that for a fully adiabatic phase change the material can cross

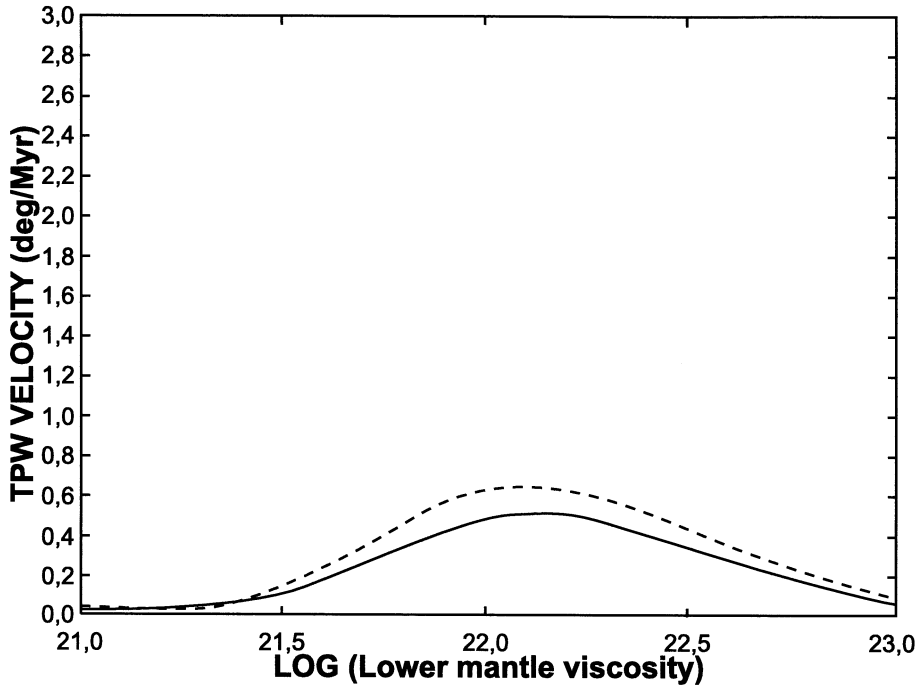


Figure 4.19. Present-day TPW as a function of lower mantle viscosity for the 5-layer earth model (Table 2.2). The solid curve corresponds to $\xi = 1.0$, while the results of the dashed one correspond to the case in which the buoyancy modes excited by the 420 and 670 km interfaces for chemical boundary conditions have been deleted. The upper mantle viscosity is fixed at 10^{21} Pa s. Redrawn from Figure 7 in Sabadini *et al.* (2002).

the boundary without experiencing any resistive or buoyant restoring force. This result demonstrates the correctness of previous efforts to reproduce the behavior of an adiabatic interface in Earth rotation models by deleting the buoyancy modes (Sabadini *et al.*, 1990; Ricard *et al.*, 1992; Mitrovica and Milne, 1998).

Figure 4.20 explains the effects of a variation in the response factor for a 5-layer model, from $\xi = 0.0$, dotted curve, to $\xi = 0.99$, solid, through intermediate values of 0.7, dashed, and 0.9, dash-dotted. These results show the continuous decay of the TPW signature for lower mantle viscosities of the order of 10^{21} Pa s from the high value of 3 Deg/Myr, dotted, corresponding to the chemically stratified model, to the value of 0.2 Deg/Myr of the adiabatic model, solid. For higher viscosities of 10^{22} Pa s, the reduction is only from 0.8 to 0.5 Deg/Myr for this range of ξ values.

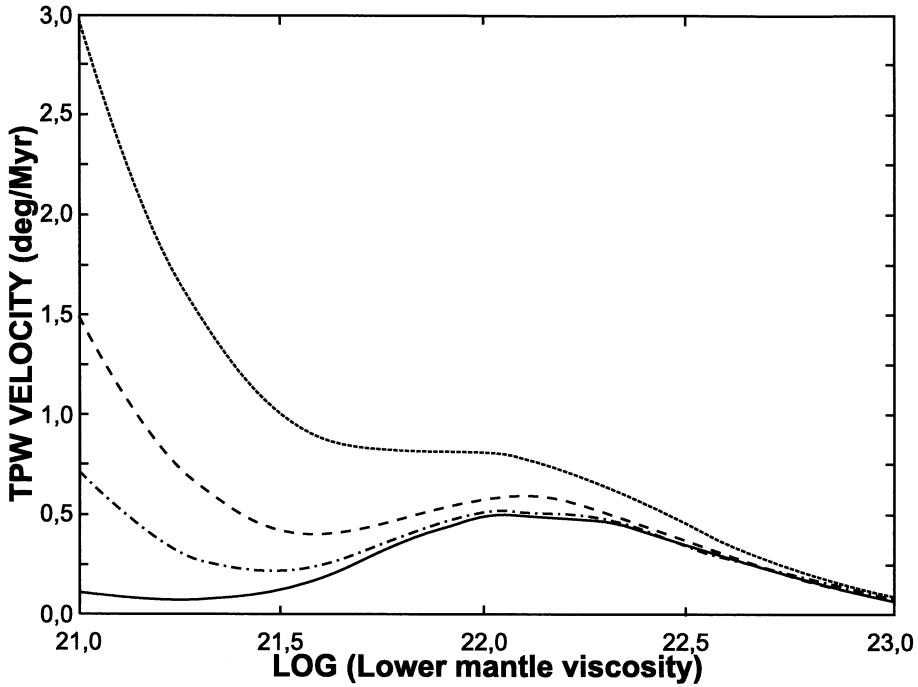


Figure 4.20. Present-day TPW as a function of lower mantle viscosity for the 5-layer earth model (Table 2.2), and phase-change boundary conditions. The dotted curve corresponds to $\xi = 0.0$, the dashed one to $\xi = 0.7$, the dash-dotted one to $\xi = 0.9$ and the solid one to $\xi = 0.99$. The upper mantle viscosity is fixed at 10^{21} Pa s. Redrawn from Figure 8 in Sabadini *et al.* (2002).

8. TRADE-OFF BETWEEN MANTLE VISCOSITY AND DENSITY CONTRASTS

The sensitivity of the TPW signal to the style of mantle density stratification for lower mantle viscosity ν_2 lower than 10^{22} Pa s obtained in the previous section poses the question of the impact of adiabaticity on lower mantle viscosity inference with respect to chemical stratification, once the observational datum of 0.9 ± 0.02 Deg/Myr (McCarty and Luzum, 1996) is considered as in section 4.4. Inspection of Figures 4.17 - 4.20 shows that changes in the response factor ξ can be more effective than changes in the lower mantle viscosity and that the variabilities in the TPW values occur in the viscosity region where the TPW curves cross the observational datum of 0.9 ± 0.02 Deg/Myr, which indicates that modifications from chemical to phase-change boundary conditions at internal interfaces have severe consequences on the estimate of lower mantle viscosity. In particular, it is remarkable that for the 31-layer model given by the dotted

curve in Figure 4.18, compatible with the PREM model, thin solid curve in Figure 4.15, the predicted TPW is always lower than 0.9 Deg/Myr. The effects of adiabaticity in the mantle and of phase changes in the transition zone are to reduce the TPW signal induced by post glacial rebound, implying that other mechanisms must be invoked to contribute to the observed TPW.

In order to elucidate the differences between the chemically stratified mantle, as used so far, and the fully adiabatic phase-change mantle, we portray in Figure 4.21 the following χ^2 analysis as a function of the upper and lower mantle viscosities

$$\chi = \frac{(\dot{m}^{mod} - \dot{m}^{obs})}{\sigma}, \quad (4.4)$$

where $\dot{m}^{mod}$ denotes the modeled polar wander velocity. We have used $\dot{m}^{obs} = 0.9$ Deg/Myr and the standard deviation $\sigma = 0.02$ Deg/Myr.

These TPW predictions are based on the 31-layer model, fully adiabatic in the uppermost mantle and lower mantle ($\xi = 1$), with $\xi = 0.7$ in the transition zone (dotted curve in Figure 4.18). With respect to the 5-layer model described in Table 4.3 and portrayed by the dash-dotted curve in Figure 4.18, this 31-layer model has the disadvantage of overestimating the effects of the possible adiabatic phase changes away from the transition zone, but has the advantage of fulfilling the requirement of an initial self-compressed state in agreement with PREM. This phase change model represents an improvement with respect to the chemically stratified mantle considered until now since, as shown in Figure 4.18, it does not overestimate the buoyancy for viscosities of the order of 10^{21} Pa s.

The upper mantle viscosity varies in the range 20.5 - 21.0 in logarithmic scale, in agreement with plausible values for this parameter, and with the findings in section 4.4. The regions where χ^2 obtains the local minima give the best fit between the modeled and observed value. The χ^2 pattern for the 31-layer model under study, in the upper mantle viscosity range 20.5 - 21.0, is substantially different from that depicted by the chemically stratified mantle, which portrays a single, well defined minimum at 21.6 due to the single crossing of the TPW curve with the observation, as shown in section 4.4. Phase changes in the transition zone and the fully adiabatic uppermost mantle and lower mantle have the effect of splitting that single minimum into two local minima, dark grey, characterized by a higher value of χ^2 , the modeled TPW being always lower than the observed value. The two minima occur for lower mantle viscosities in proximity of 10^{21} Pa s and 10^{22} Pa s, corresponding to the highest TPW signal and close to the observed value.

The first minimum at 10^{21} Pa s is due to the Earth's rotation instability driven by sustained mantle flow; rotational disequilibrium is maintained by the buoyancy at the transition zone. The second minimum is due to a global situation

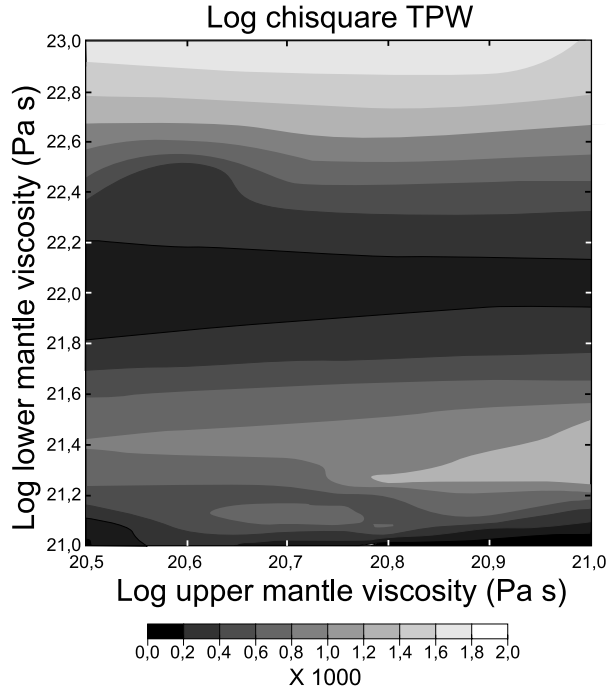


Figure 4.21. Logarithm of χ^2 plotted as a function of the upper and lower mantle viscosity when the TPW data by McCarthy and Luzum (1996) are considered. Redrawn from Figure 12 in Sabadini *et al.* (2002).

of isostatic disequilibrium. This physical interpretation of the two minima in the TPW is identical to that commonly used to explain the two viscosity solutions for the time derivative of the second degree harmonic component of the geopotential $\dot{J}_2$ (Mitrovica and Peltier, 1993). An adiabatic mantle, rather than a chemical one, facilitates the attainment of global rotational equilibrium because the material has the ability to cross the interfaces of different density; this ‘easy crossing’ allows present-day high flow rates in the mantle, as shown by the low viscosity minimum.

The opposite is true for the second minimum, with inhibited mantle flow due to the transition zone, which maintains a global rotational disequilibrium.

The relatively high χ^2 values of the two minima, which reflect the low TPW rates for the 31-layer adiabatic mantle model, has an important implication on present-day ice mass instabilities over the polar regions of the Earth, in particular in Antarctica and Greenland, since it is clear, on the basis of our results, that postglacial rebound can hardly explain the whole amount of the observed 0.9 ± 0.02 Deg/Myr: TPW driven by post-glacial rebound is thus barely enough to explain the present-day drift of about 1 Deg/Myr, and suggests

that extra sources of TPW, eventually due to ice mass loss in Antarctica and Greenland, should be invoked to account for the observation.

Figure 4.21 portrays another feature that characterizes the TPW calculations, namely the low sensitivity of TPW to upper mantle viscosity variations, as already shown in section 4.4.2. This is on the other hand expected since the Earth's rotation is dependent on the global deformation of the planet, where the lower mantle plays a dominant role due to its larger volume with respect to the upper mantle.

9. IMPACT OF MANTLE STRATIFICATION ON TPW DURING THE ICE AGES

We now consider the TPW pattern over the complete ice-age period reflected in Figure 4.2. As shown by equation (3.61), the displacement of the axis of rotation during the active phase of loading depends not only on the viscous relaxation terms with the poles in $s = a_i$, but also on the pole in $s = 0$, which provides the secular term, as discussed in section 3.5 on rotation dynamics.

To gain insight into the physics of Earth rotation during the active phase of loading, we start in Figure 4.22 with a simple earth model, not containing the 670 km discontinuity, characterized by a uniform mantle density ($\rho = 4314 \text{ kg/m}^3$) and viscosity ($\nu_1 = 10^{21} \text{ Pa s}$). The solid curve accounts for both the $s = 0$ and $s = a_i$ poles, while the dashed one does not contain the contribution from $s = 0$, namely $A_0^* = 0$. The solid curve shows that the axis of rotation oscillates in phase with the glacial cycles around a mean position that increases in time in a linear fashion. The $s = 0$ pole is clearly the major contributor to the net shift of the axis of rotation after the end of the last deglaciation, as indicated by the constant horizontal line. This is confirmed by the dashed curve at the bottom of this figure, which corresponds to contributions arising solely from the $s = a_i$ poles. These poles do not contribute to the net shift of the axis of rotation during the continuous occurrence of ice ages. Thus, in the absence of density stratification in the mantle, the net shift of the axis of rotation is totally due to the $s = 0$ pole. Furthermore, the velocity of polar wander averaged over the complete period of ice loading can be very high, about 1 deg/Myr , at least for a mantle viscosity of 10^{21} Pa s (Sabadini *et al.*, 1982b, 1983; Sabadini and Vermeersen, 1997b).

If the viscosity is increased, the velocity of polar wander is reduced accordingly. Indeed, the $s = 0$ pole contributes a large net shift of the axis of rotation only if the mantle viscosity is about 10^{21} Pa s . The relaxation terms associated with the $s = a_i$ poles do not play any role in the net shift, due to the fast relaxation modes a_i . If the mantle is not stratified in density, large excursions of the rotation pole can be obtained.

The physics underlying the $s = 0$ pole is evident in equation (3.64) where the loading part yields $(1 + k_f^L)$. The latter quantifies the amount of isostatic

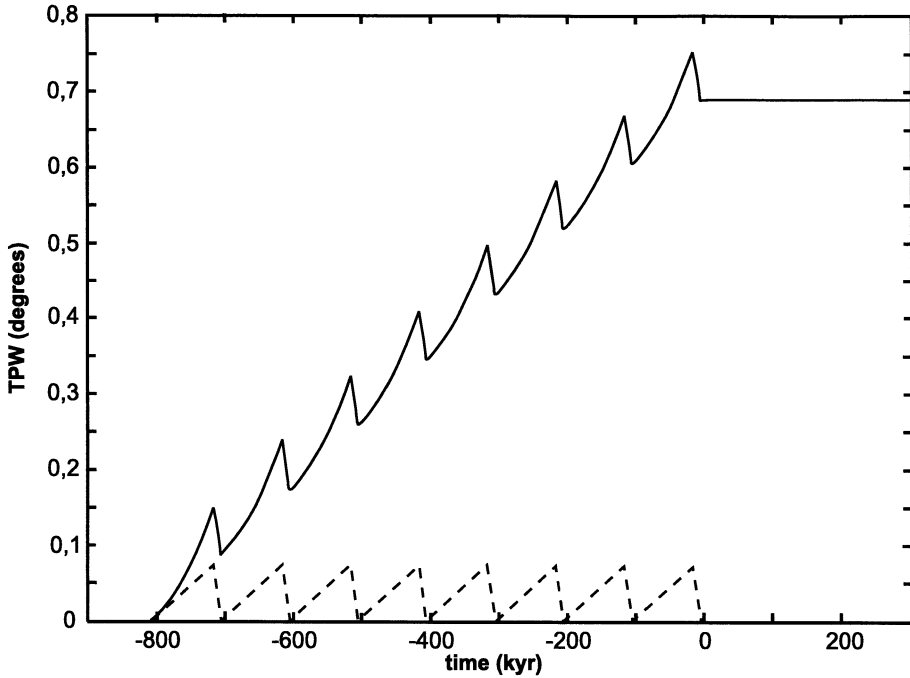


Figure 4.22. TPW displacement during eight cycles of glaciation-deglaciation, and 3×10^5 yr after the end of deglaciation at $t = 0$, for a 3-layer model consisting of an elastic lithosphere, a uniform viscoelastic mantle of 10^{21} Pa s and a core. All the poles in $s = 0$ and $s = a_i$ are considered for the solid curve, while the dashed one corresponds to $A_0^* = 0$. Redrawn from Figure 9 in Sabadini *et al.* (2002).

disequilibrium after complete relaxation has taken place. The $(1 + k_f^L)$ factor depends on the elastic properties of the lithosphere, and would be zero if the lithosphere is absent. The term $A_0^* = A_0(1 + k_f^L)$ shows that the contribution arising from the $s = 0$ pole depends linearly on the load, which is not isostatically compensated. The contribution from the $s = 0$ pole after the end of the last ice age is identical to that produced by a constant load that stands on the Earth's surface for a span of time corresponding to the eight cycles, with an ice mass which is one-half of the ice mass at the ice maximum, as indicated by the $1/2$ factor multiplying the term A_0^* in equation (4.2).

Figure 4.23 deals with the 5-layer, volume-averaged model in Table 2.2, characterized by a chemical density contrast at 420 and 670 km. In comparison with Figure 4.22, the noticeable effect of density stratification is a reduction of the net shift of the rotation axis after the eight glacial cycles and the appearance of an exponential behavior in the mean rotational response of the Earth, as already shown by Sabadini and Vermeersen (1997b). The most interesting

result in Figure 4.23, in comparison to Figure 4.22, involves the substantial reduction of the contribution from the $s = 0$ pole on the net shift of the pole. Comparison between the dashed and solid curves, corresponding to the cases in which the $s = 0$ pole is included (dashed) or not included (solid), shows that, in contrast with the previous case of a uniform mantle, the $s = a_i$ poles play a major role in the shift of the rotation axis. The density stratification of the mantle is responsible for a stabilization of the Earth's rotational state; in particular its rotation behavior is controlled by the density contrasts occurring in the transition zone. Density stratification in the transition zone introduces in fact the long rotational relaxation modes depicted in Figure 4.16, circles and crosses.

This decrease in the contribution from the $s = 0$ pole is accompanied by an increase in the contribution from the a_i modes, corresponding to the slowest relaxation rotational modes (which are associated with the density contrasts of the transition zone). This slower relaxation is responsible for the exponential behavior of the net shift of the axis of rotation. It is thus clear that the appearance of the slow relaxation rotational modes in the 5-layer model is responsible for the increase in the strength γ_i/a_i relative to the A_0^* contribution, which causes a substantial modification of the pattern of the polar wander curves, from a linear growth to an exponential relaxation. The net shift is due to the fact that after an ice cycle has been completed the Earth is still out of isostatic equilibrium. The relaxation times of some of the modes by which the Earth relaxes to the changed surface mass distribution are in fact longer than the periods of the ice cycles. Comparison between Figures 4.22 and 4.23 indicates that, since the Earth is chemically stratified, the realistic pattern of polar wander is that portrayed by the 5-layer model, where the dominant role is played by the $s = a_i$ poles. This has important implications for the net shift of the axis of rotation that the Earth can gain during a finite series of ice ages. The shift that is acquired permanently is that associated with the $s = 0$ pole, while that due to relaxation is recovered after the series of cycles once a sufficiently long span of time has elapsed. This span of time is controlled by the slowest rotational mode, as shown by the part of the curve in Figure 4.23 after the end of active glaciation phases. After the last glacial event the axis of rotation comes back to a position close to the initial one before the beginning of the first cycle, with an offset due to the net shift gained by the pole in $s = 0$. These results elucidate the major impact of density stratification on the asymptotic behavior of the rotation equations, with a reduction of a factor 14 from the 0.7° net shift of the 3-layer model to 0.05° of the 5-layer one after complete relaxation, which takes several million years.

Within the framework of the 5-layer model, Figure 4.24 shows the effects of the phase change at 420 and 670 km, solid curves, compared to the chemical transition zone, dashed. This figure clearly shows the effects of the increase in the rotational relaxation times, responsible for a slower decay of the displace-

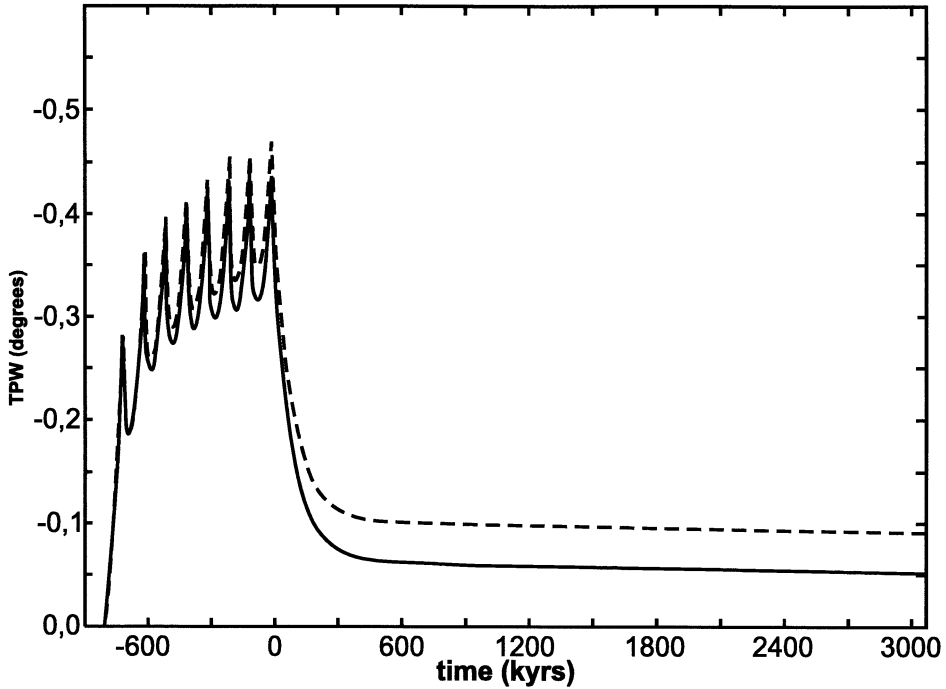


Figure 4.23. TPW displacement during eight cycles of glaciation-deglaciation, and 3×10^6 yr after the end of deglaciation at $t = 0$, for a 5-layer, chemically stratified model (Table 2.2). The upper and lower mantle viscosity is fixed at 10^{21} Pa s. All the poles in $s = 0$ and $s = a_i$ are considered for the dashed curve, while the solid one corresponds to $A_0^* = 0$. Redrawn from Figure 10 in Sabadini *et al.* (2002).

ment after the termination of the ice cycles, and of a substantial increase of fifty per cent during the active phase of glaciation. Since the Earth carries a transition zone, ice loading can be responsible for a conspicuous shift of the axis of rotation only during active phases of glaciation. The drastic reduction in the net shift of the rotation axis for a realistic earth model is due to the physics of two different immiscible viscous fluids filling the upper and lower mantle, which inhibits the readjustment of the equatorial bulge during rotation deformation. The transition zone, irrespective of being chemical or phase change, acts like a barrier, an impenetrable one for the chemical case and a leaky one for the phase change, that dampens the ability of the Earth to wander due to the continuous occurrence of ice ages, but at the same time maintains the Earth rotationally unstable for present day TPW. A fully adiabatic lower mantle and upper mantle, in conjunction with a transition zone where density increase occurs via phase changes, are responsible for a substantial reduction in the TPW with respect to fully non-adiabatic mantle models considered in previous studies, for viscosities of about 10^{21} Pa s, in which the rotational instability is mainly due to the

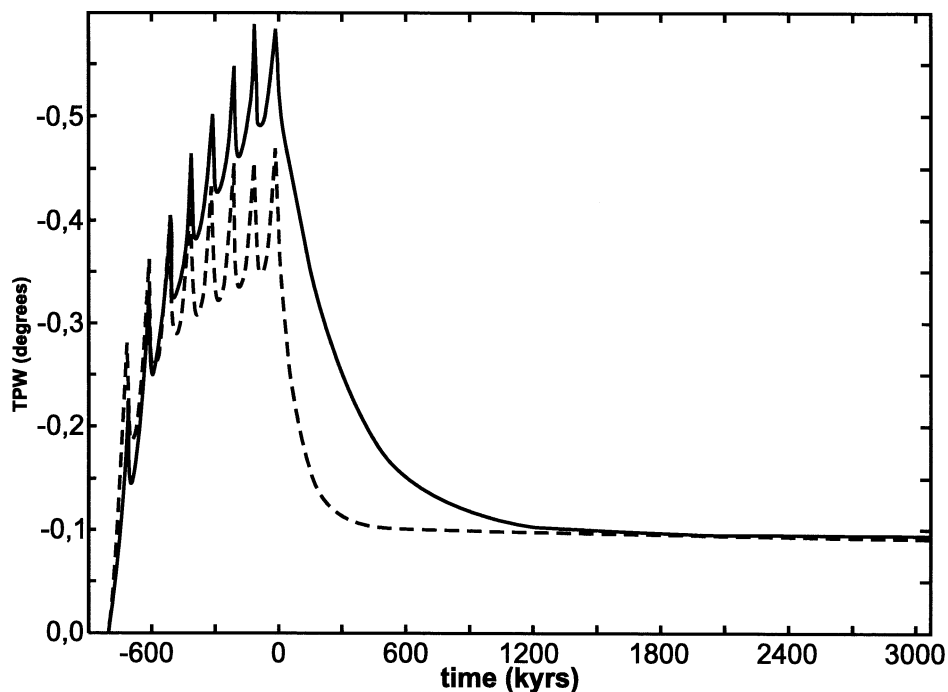


Figure 4.24. TPW displacement during eight cycles of glaciation-deglaciation, and 3×10^6 yr after the end of deglaciation at $t = 0$, for a 5-layer (Table 2.2), phase-change model with $\xi = 0.7$ (solid). The dashed curve corresponds to the chemically stratified mantle in Figure 4.23. The upper and lower mantle viscosity is fixed at 10^{21} Pa s. Redrawn from Figure 11 in Sabadini *et al.* (2002).

high flow rates in the mantle. The usage of chemically stratified models has thus led to a substantial overestimation of the TPW due to the last ice age.

Chapter 5

DETECTION OF THE TIME-DEPENDENT GRAVITY FIELD AND GLOBAL CHANGE

1. CHANGES IN THE LONG-WAVELENGTH GEOID COMPONENTS FROM SATELLITE LASER RANGING TECHNIQUES

In the dynamic Satellite Laser Ranging (SLR) data analysis approach, the satellite's orbit can be used as a gravity probe to monitor the time-varying gravity field. A long history of SLR observations of the geodetic satellites LAGEOS-I, LAGEOS-II, Starlette and Stella have been analyzed by Devoti *et al.* (2001) to estimate the time series of the low degree zonal coefficients in the Earth's gravity field and derive their secular drifts up to degree 6. Comparison of these zonal rates with the results of the viscoelastic earth models in Chapters 1 - 2, forced by Pleistocene deglaciation, shows that the SLR-retrieved even and odd zonals can be used today to infer mantle viscosity and lithospheric thickness. Following Mitrovica and Peltier (1989), the time derivatives of the zonals are defined as $\dot{J}_\ell = -\frac{1}{a}\sqrt{2\ell+1}\dot{G}_{\ell 0}(t)$, where $G_{\ell 0}$ are the zonal components of the geoid anomaly due to ice loading and sea water redistribution defined in equations (6.12) and (6.14) in the following Chapter 6 on sea level changes; a denotes the Earth's radius and the dot the differentiation with respect to time. In this Chapter 5, we will consider the effects on $\dot{J}_\ell$ due to both the deglaciation of the Pleistocene and present-day ice loss in Antarctica and Greenland.

In Devoti *et al.* (2001) the observations of four geodetic satellites are analyzed for the period 1987 - 1998, spanning 11 years of LAGEOS-1 and Starlette observations and 5 and 4 years of LAGEOS-2 and Stella observations. Unmodeled and mismodeled tidal and non-gravitational perturbations could in principle affect the long-term gravity signal and may partly explain the discrepancies between the time derivatives of the zonals given in Tables 5.1 and 5.2, taken from

Table 5.1. Even degree zonal secular drift estimated by different authors, units: 10^{-11} /year

Author	$\dot{J}_2$	$\dot{J}_4$	$\dot{J}_6$
Yoder <i>et al.</i> (1983)	-3		
Rubincam (1984)	-2.6 ± 0.6		
Cheng <i>et al.</i> (1989)	-2.5 ± 0.3	0.3 ± 0.6	
Gegout and Cazenave (1993)	-2.8 ± 0.4		
Eanes (1995)	-2.4 ± 0.2		
Nerem and Klosko (1996)	-2.8 ± 0.3	0.2 ± 1.5	
Cazenave <i>et al.</i> (1996)	-3.0 ± 0.5	-0.8 ± 1.5	
Cheng <i>et al.</i> (1997)	-2.7 ± 0.4	-1.4 ± 1.0	0.3 ± 0.7
Devoti <i>et al.</i> (2001)	-2.9 ± 0.2	0.6 ± 0.5	0.3 ± 0.3

Table 5.2. Odd degree zonal secular drift estimated by different authors, units: 10^{-11} /year

Author	$\dot{J}_3$	$\dot{J}_5$	$\dot{J}_{odd}$ ($\dot{J}_3 + 0.837\dot{J}_5$)	$\dot{J}_{odd}$ ($\dot{J}_3 + 0.9\dot{J}_5$)
Nerem and Klosko (1996)			1.6 ± 0.4	
Cheng <i>et al.</i> (1997)	-1.3 ± 0.5	2.1 ± 0.6	0.5	0.6
Devoti <i>et al.</i> (2001)				0.5 ± 0.2

Devoti *et al.* (2001), and recent published results (Eanes, 1995; Cazenave *et al.*, 1996; Nerem and Klosko, 1996; Cheng *et al.*, 1997).

Another critical point in treating a long period of tracking data arises from the lack of homogeneity in data quality and distribution. Devoti *et al.* (2001) provide a detailed description of the techniques applied to recover the zonals given in Table 5.1 and 5.2. Figure 5.1 shows the characteristics of the satellites which have been used to retrieve the SLR solution analyzed in this chapter. Figure 5.2 shows the MLRO (Matera Laser Ranging Observatory) of the Center of Space Geodesy "G. Colombo" in Matera (Italy) (Italian Space Agency, ASI) for SLR analyses.

In order to explore the capability of our SLR-retrieved zonals to infer the upper mantle viscosity ν_2 and the lithospheric thickness L , we model the viscoelastic response of our planet to Pleistocene deglaciation, the major contributor to the zonal rates given in Tables 5.1 and 5.2; the ice model is the ICE-3G by Tushingham and Peltier (1991). In this section, present-day forcing is not included in the modeling, and we thus assume that Pleistocene deglaciation is the only forcing contributor. We can look for discrepancies in the results of the modeling as possible indicators for present-day ice melt or growth. In Figures 5.3 and 5.4 a χ^2 analysis is carried out for each of the zonals in Tables 5.1 and 5.2 estimated by Devoti *et al.* (2001) in order to explore their preferred lower and upper mantle viscosities, ν_2 and ν_1 , and lithospheric thickness, L ,

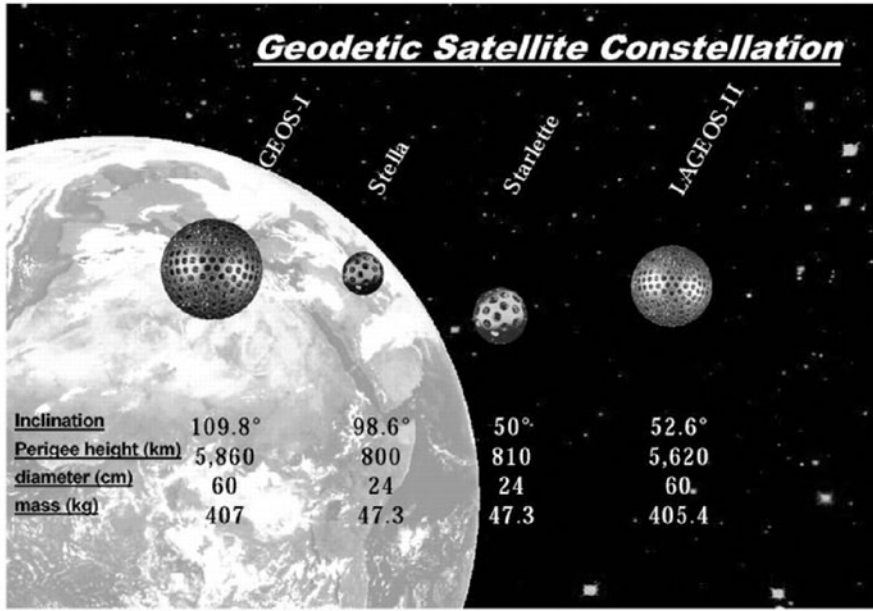


Figure 5.1. Satellites for laser tracking used in the SLR analyses of this chapter, with the Earth in the background.

and to elucidate possible discrepancies among these best-fit parameters. The χ^2 analysis for each zonal separately is given by

$$\chi^2 = \left(\frac{j_l^{mod} - j_l^{obs}}{\sigma_l^{obs}} \right)^2, \quad (5.1)$$

where j_{obs} and j_{mod} correspond to the zonal in the tables and that retrieved from the models, respectively; σ_l represents the standard deviation of the observed value.

The regions where the χ^2 attain their (local) minima give the best fit between observed and modeled values. In agreement with Vermeersen *et al.* (1998), where, except for the odd one, the χ^2 analysis was carried out for all the zonals simultaneously or the zonal plus TPW, the best fit in Figure 5.3 falls into two main regions, one in which the lower mantle viscosity is high, of the order of 10^{23} Pa s, and another one in which it is of the order of 10^{21} Pa s or lower.

Figure 5.3 shows that the tendency of this multiplicity of solutions is a characteristic of each zonal. The best-fit ν_2 varies among the zonals, with a clear tendency for j_{odd} , j_4 and j_6 towards lower mantle viscosities that fall beyond the limits stated above, with $\nu_2 < 10^{21}$ Pa s and $\nu_2 > 10^{23}$ Pa s. By contrast,

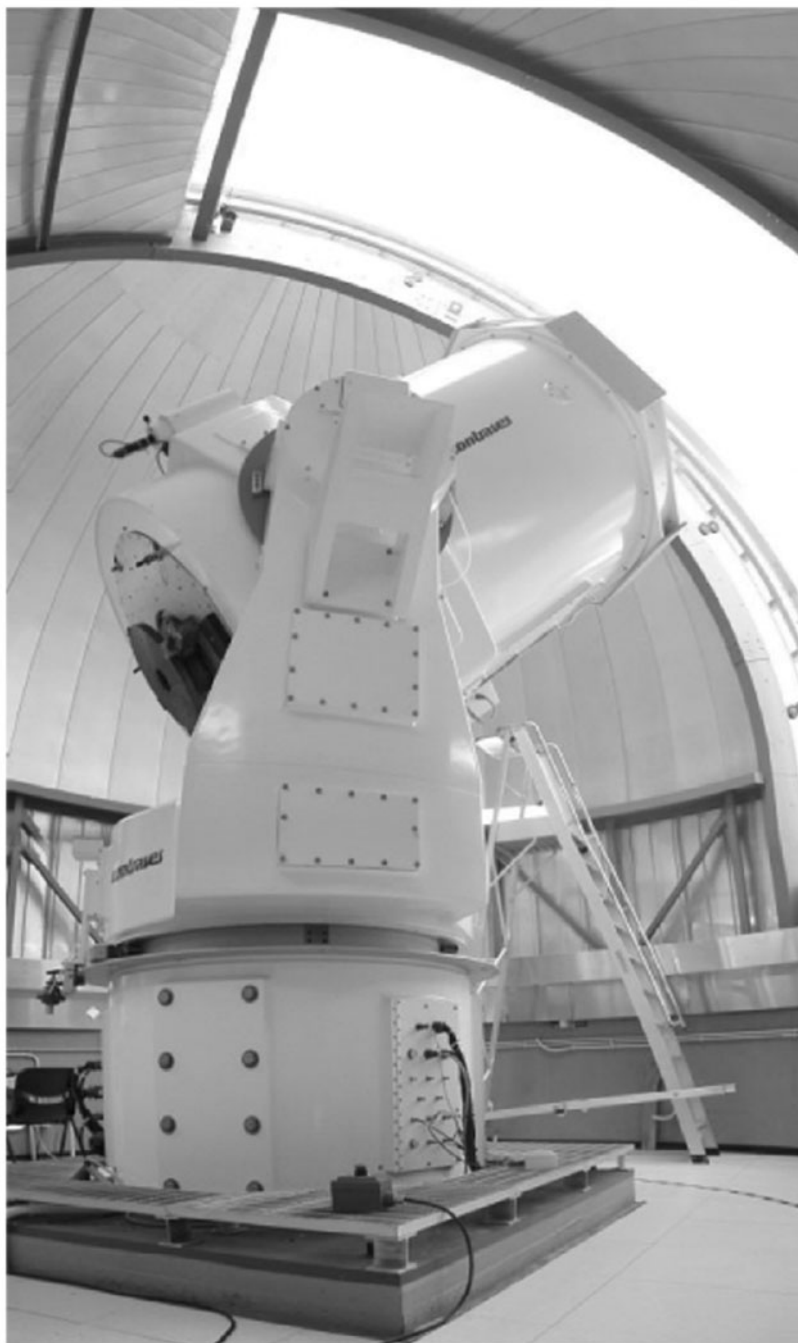


Figure 5.2. MLRO (Matera Laser Ranging Observatory) of the Center of Space Geodesy "G. Colombo" in Matera, Italy (Italian Space Agency, ASI.)

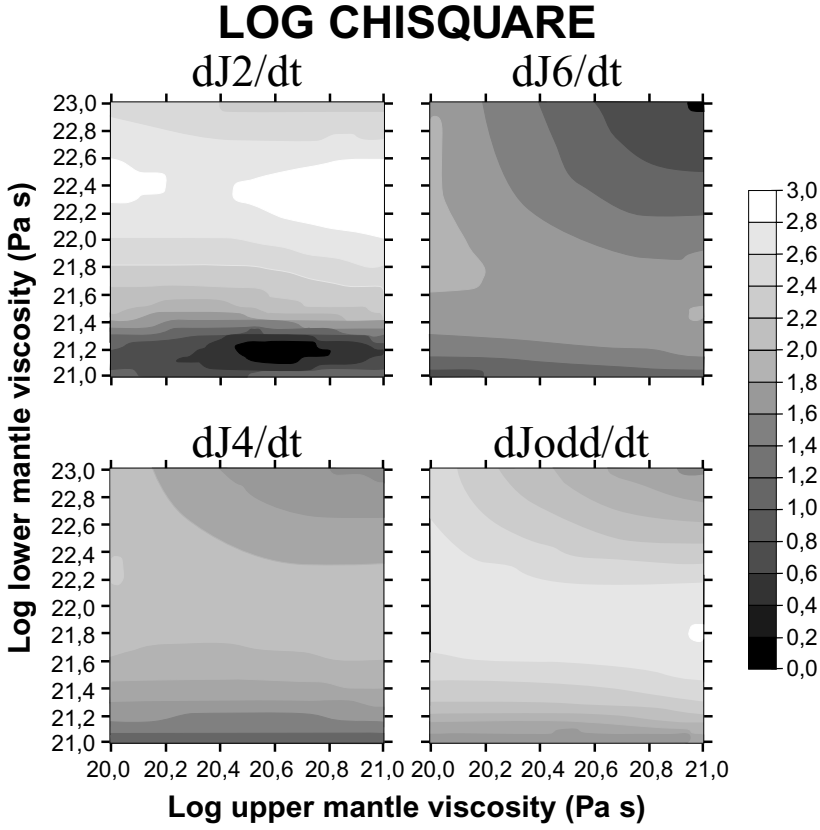


Figure 5.3. χ^2 analysis for the time derivatives of the zonals given in Tables 5.1 and 5.2 for varying upper and lower mantle viscosities.

the best-fit lower mantle viscosity for J_2 is 21.2 in the logarithmic scale. This discrepancy between $\dot{J}_2$ and the other zonals indicates that, beyond post-glacial rebound, another mechanism of mass redistribution must be active. Another interesting result in Figure 5.3 is that the best-fit upper mantle viscosity is lower than 10^{21} Pa s, for $\nu_2 = 10^{21}$; in particular, $\nu_1 = 10^{20.6}$ Pa s for $\dot{J}_2$. For the best-fit corresponding to $\nu_2 > 10^{23}$ Pa s, it is interesting to note that the upper mantle viscosity should be higher than 10^{21} Pa s.

These results indicate an upper mantle viscosity considerably lower than 10^{21} Pa s for a lower mantle viscosity of the order of $10^{21} - 10^{22}$ Pa s, in good agreement with previous findings (Nakada and Lambeck, 1989; Forte and Mitrovica, 1996). The zonal analysis carried out in Figure 5.3 shows that the sensitivity to lower mantle viscosity variation is highest for $\dot{J}_2$ and $\dot{J}_{odd}$, lowest for $\dot{J}_6$ and somehow intermediate for $\dot{J}_4$. The highest sensitivity to upper

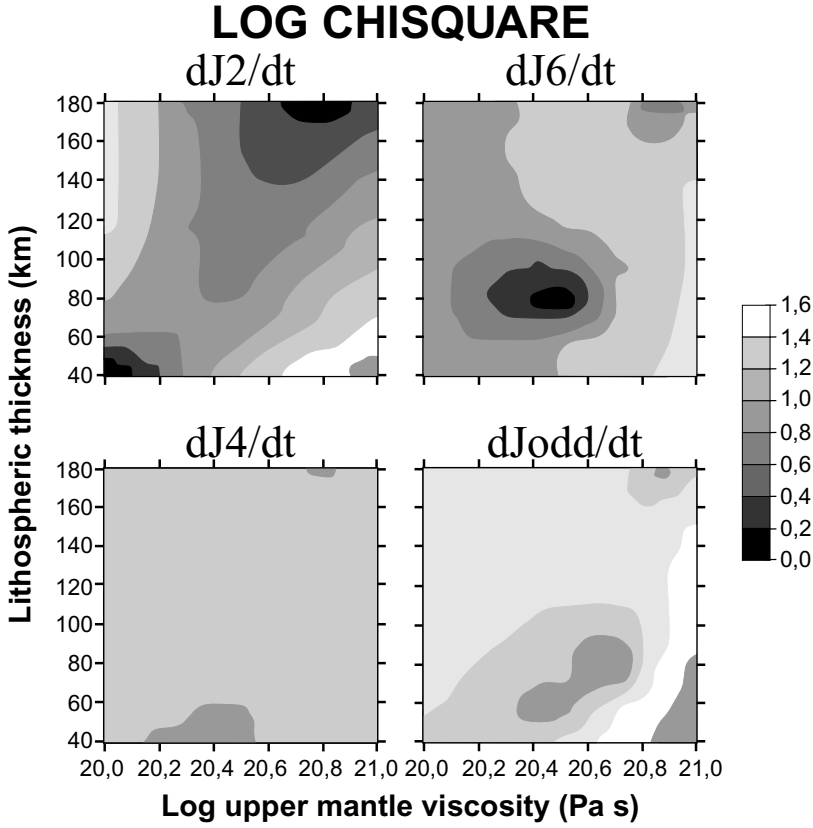


Figure 5.4. χ^2 analysis for the time derivatives of the zonals given in Tables 5.1 and 5.2 for varying upper mantle viscosity and lithospheric thickness.

mantle viscosity variations is portrayed by $\dot{J}_2$ for ν_2 of the order of 10^{21} Pa s, and by the other zonals for ν_2 of the order of 10^{23} Pa s.

The same χ^2 analysis is carried out in Figure 5.4 for lithospheric thickness and upper mantle viscosity. As observed in Figure 5.3, the preferred upper mantle viscosity is generally lower than 10^{21} Pa s. In particular, a local minimum in $\dot{J}_6$ corresponds to an upper mantle viscosity of 20.4 in logarithmic scale.

Consistent with $\dot{J}_6$, upper mantle viscosities lower than 10^{21} Pa s are indicated by $\dot{J}_{odd}$. This tendency to upper mantle viscosities lower than 10^{21} Pa s is portrayed even by $\dot{J}_2$ and $\dot{J}_4$, although the sensitivity to upper mantle viscosity variations is lowest for $\dot{J}_4$. When we consider the best-fit lithospheric thickness, it is remarkable that $\dot{J}_4$ and $\dot{J}_6$ are sensitive to lithospheric thickness variations and coherently indicate a lithosphere of 80-100 km. This finding is in contrast with $\dot{J}_2$, where local minima indicate lithospheric thickness lower than 40 km or

higher than 180 km. This discrepancy could also be interpreted as an indication of ongoing mass redistribution over the Earth. A lithosphere thicker than 180 km is consistent with the inference obtained in section 4.5 from I.L.S and l.o.d. data.

Some interesting observations arise if we compare the viscosity values that fit the various harmonics or we test the mutual consistency of the different SLR analyses in Tables 5.1 and 5.2. For an upper mantle viscosity of 10^{21} Pa s, $\dot{J}_2$ can be fitted by a lower mantle viscosity of 1.5×10^{21} Pa s, while $\dot{J}_{odd}$ in this section or in Cheng *et al.* (1997) requires a lower mantle viscosity of $2 - 6 \times 10^{20}$ Pa s; the datum in Nerem and Klosko (1996) for $\dot{J}_{odd}$ requires 1.0×10^{21} Pa s. The $\dot{J}_4$ from Nerem and Klosko (1996) and the SLR analysis carried out in this section agree within 2σ , but the sign is opposite with respect to the model predictions. Our modeling, with a mantle viscosity of 10^{21} Pa s, agrees in amplitude and sign with Cheng *et al.* (1997) but only in amplitude with the SLR analysis by Devoti *et al.* (2001). For $\dot{J}_6$, our SLR analysis agrees with Cheng *et al.* (1997) and is coherent with our model predictions as far as the amplitude is concerned, but is opposite in sign. These findings show that discrepancies still exist between model predictions and the various SLR analyses, as well as among the SLR analyses themselves.

These results show that SLR-retrieved low degree geoid components J_2 and J_{odd} can be used in principle to constrain the rheology of the mantle and lithospheric thickness. Discrepancies in the viscosity values needed to fit the zonal rates when Pleistocene deglaciation is the only forcing mechanism are a strong indication that mass redistribution is actually occurring over the Earth, eventually being associated with mass instabilities in Greenland and Antarctica, as also suggested by Johnston and Lambeck (1999): these arguments are explored in the following section.

2. TRADE-OFF BETWEEN LOWER MANTLE VISCOSITY AND PRESENT-DAY MASS IMBALANCE IN ANTARCTICA AND GREENLAND

Redistribution of mass in the Earth modifies the gravitational field and causes perturbations in the moment of inertia, which in turn are responsible for a drift of the rotation axis (TPW). Two major mechanisms are responsible for the secular changes in the gravitational field and rotation: the Pleistocene deglaciation, as shown in Figure 4.2, and the present-day mass instability in Antarctica, Greenland and Alpine glaciers. The Earth is affected today by the first mechanism because of the viscous memory of the mantle and by the second due to ongoing surface mass redistribution.

In the previous section, comparison between SLR-retrieved zonals and predictions from viscoelastic models driven solely by Pleistocene deglaciation showed that this mechanism cannot be the only source of time variations of the

gravity field (Devoti *et al.*, 2001). In fact, discrepancies in the viscosity profiles required to reproduce the different zonals when the only forcing mechanism is postglacial rebound seem to indicate ongoing mass redistribution over the Earth as related to global change and eventually associated with mass instabilities in Antarctica and Greenland. Both the even and odd zonal geopotential components of the gravity field, up to harmonic degree $l = 6$, are used in this section in conjunction with the modeled ones to infer the key parameters that control the two major mechanisms above: the viscosity, characterizing the flow properties of the mantle, and the ice mass imbalance in Antarctica and Greenland, which are thought to be the contributors to ongoing mass redistribution.

The $\dot{J}_{odd}$ contribution by Devoti *et al.* (2001) in Table 5.2 can be better estimated by the following expression appropriate for a single satellite solution (Starlette) (Schutz *et al.*, 1993)

$$\begin{aligned} \dot{J}_{odd} = & \dot{J}_3 + 1.04\dot{J}_5 - 0.53\dot{J}_7 - 0.81\dot{J}_9 + 0.13\dot{J}_{11} \\ & + 0.52\dot{J}_{13} + 0.06\dot{J}_{15} - 0.3\dot{J}_{17} - 0.11\dot{J}_{19} + 0.14\dot{J}_{21}. \end{aligned} \quad (5.2)$$

The highest odd zonals are not considered since they would enter equation (5.2) with coefficients smaller than 0.1. Except for $\dot{J}_{odd}$ of equation (5.2), we will assume in our study that all the other zonals are sufficiently well separated so as to draw sound geophysical conclusions. In this section, we make use of the zonal secular drifts of Cheng *et al.* (1997) and Devoti *et al.* (2001) given in Tables 5.1 and 5.2, except for $\dot{J}_{odd}$ in Devoti *et al.* (2001) now based on equation (5.2). These data are used to constrain the upper and lower mantle viscosities and the amount of mass imbalance in Antarctica and Greenland by means of the χ^2 analysis for each zonal separately, as discussed in the previous section, in order to emphasize that the simultaneous fit of all the zonals can occur only for a specific range of lower mantle viscosity values and ice mass imbalance, which allows the minima in χ^2 for the different $\dot{J}_l$ to overlap in the viscosity space.

Ice mass imbalance in Antarctica and Greenland is varied until the best-fit lower mantle viscosities obtained from equation (5.1) are the same for all the zonals. Ice mass in Antarctica is varied in our modeling within the bounds of -500 to $+400$ Gt/yr, as per previous observations of grounded ice (Bentley and Giovinetto, 1991; Jacobs, 1992; Warrick *et al.*, 1996). For Greenland we have considered the case of ice loss from 0 Gt/yr to -144 Gt/yr, the latter corresponding to a sea-level rise of 0.4 mm/yr for a 1°C warming (Oerlemans, 1991). The viscoelastic part of the earth model is only needed for the Pleistocene deglaciation, while present-day mass imbalance in Antarctica and Greenland requires the elastic component. A better physical insight into the results of the viscosity inversion can be gained if we show first the dependence of the $\dot{J}_l$ on the lower mantle viscosity and on the rate of melting in the polar regions. In Figure

5.5 the $\dot{J}_l$ are shown as a function of lower mantle viscosity for Pleistocene (solid), Pleistocene plus ice loss in Antarctica of -500 Gt/yr (dashed), and Pleistocene plus ice loss in Antarctica and Greenland of -500 and -144 Gt/yr (dotted), with $\dot{J}_{odd}$ based on the equation (5.2). Note the sensitivity of the zonals to lower mantle viscosity variations and the dominant effect of ice loss in Antarctica with respect to Greenland. The peak value in the $\dot{J}_2$ in the solid curves for the Pleistocene deglaciation is displaced from 2×10^{22} Pa s, corresponding to $\dot{J}_2$, towards lower values of lower mantle viscosity for increasing harmonic order, in agreement with Mitrovica and Peltier (1993), down to 3×10^{21} Pa s for $\dot{J}_6$. The modeled zonals show the tendency to admit two intersections of the model results, with the horizontal stripes depicting the Cheng *et al.* (1997) and Devoti *et al.* (2001) zonal secular drifts, light and dark grey, respectively, thus providing two possible lower mantle viscosities. The lower viscosity solution corresponds to a situation of sustained flow in the mantle and a present-day configuration which is close to global post-glacial isostatic equilibrium, while the higher viscosity solution corresponds to reduced mantle flow and large isostatic disequilibrium. The solid curves show that Pleistocene forcing does not allow a simultaneous fit of the SLR data for all the zonals with the same lower mantle viscosity. The scenarios of ice mass imbalance in Antarctica and Greenland suggest that this inconsistency, already noted in the previous section, could be due to some amount of ice mass instability occurring today in these two regions but not included in the solid curves. The dashed and dotted curves show how ice loss in Antarctica and Greenland impacts the results of the Pleistocene deglaciation.

With respect to the Pleistocene solid curves, the effects of melting in Antarctica are to displace the peak values in $\dot{J}_l$ in the direction of the Cheng *et al.* (1997) and Devoti *et al.* (2001) data for the even and odd zonals, eventually exceeding the data themselves as shown by the dashed curves for $\dot{J}_6$ and the odd zonals because of the extreme value of -500 Gt/yr used in the simulations. Comparison between the dashed and dotted curves shows that ice loss in Greenland reinforces the effects of Antarctica on the even zonals and counteracts Antarctica on the odd ones (Mitrovica and Peltier, 1993; James and Ivins, 1997). These results indicate that ice growth in Antarctica must be excluded because it would cause a displacement of the Pleistocene curves in the opposite direction with respect to Cheng *et al.* (1997) and Devoti *et al.* (2001), while at the same time suggesting that present-day mass imbalance may make it possible to solve for the inconsistencies in the lower mantle viscosity inferences noted above.

In the following figures, both SLR solutions are used within a χ^2 approach, where ice mass loss in Antarctica and Greenland is added to Pleistocene deglaciation; the effects of Alpine glaciers are negligible and have not been considered. Since Antarctica is the largest contributor, we have adopted the strategy of con-

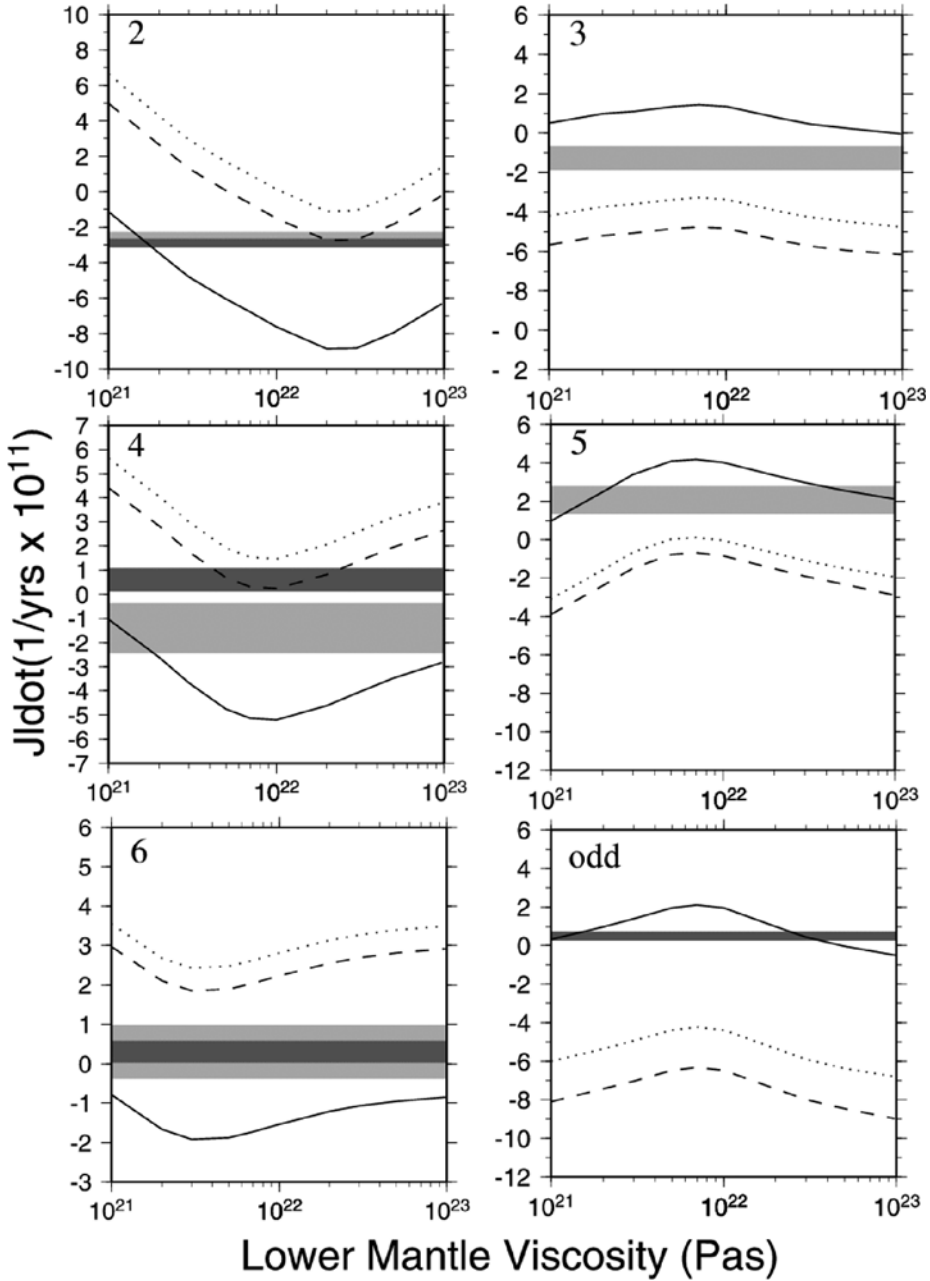


Figure 5.5. $\dot{J}_l$ as a function of lower mantle viscosity, ranging from 10^{21} to 10^{23} Pa s. The numbers in the top, left part of the panels indicate the harmonic degree or the cumulative odd degrees as in equation (5.2), in the bottom, right panel. Upper mantle viscosity is fixed at 5×10^{20} Pa s. The light and dark grey stripes stand for the Cheng *et al.* (1997) and Devoti *et al.* (2001) solutions. Solid curves correspond to Pleistocene deglaciation, dashed ones to Pleistocene plus maximum ice loss in Antarctica and the dotted ones include maximum ice loss in Greenland. The modeled results for $\dot{J}_{odd}$ stand for the combination of zonals given by equation (5.2).

straining ice loss in Antarctica first by means of Cheng *et al.* (1997), where the odd zonals are not contaminated by higher harmonic components; the effects of ice loss in Greenland on viscosity estimates are then analyzed. After having explored all the possibilities for ice loss in Antarctica between 0 Gt/yr and -500 Gt/yr, we find that the best-fit lower mantle viscosities from the different zonals in Cheng *et al.* (1997) are the same if ice mass imbalance is close to -250 Gt/yr, which will be used throughout. The left columns in Figures 5.6 and 5.7 correspond to this scenario and provide the χ^2 patterns for the zonal secular drifts of Cheng *et al.* (1997) and Devoti *et al.* (2001), respectively; melting in Greenland at the rate of -144 Gt/yr is added in the right columns. χ^2 varies as a function of upper and lower mantle viscosity, horizontal and vertical scales, respectively. The grey scale provides the χ^2 values, with the minima providing the best-fit solutions given by the darkest grey. Figure 5.6, left column, shows that for -250 Gt/yr of ice loss in Antarctica the best-fit lower mantle viscosity solutions overlap for all the zonals and that for the best resolved zonals $\dot{J}_2$, $\dot{J}_3$ and $\dot{J}_5$ the χ^2 minimum does not coincide with 21.2, corresponding to the first intersection of the $\dot{J}_2$ solid curve with SLR data, top left panel in Figure 5.5, but is displaced to higher values, 21.6-22.0 in logarithmic scale; $\dot{J}_4$ and $\dot{J}_6$ are coherent with this picture, although less resolved. A substantial change in the rate of melting in Antarctica with respect to -250 Gt/yr would not allow a simultaneous fit of the zonals with the same viscosity profile. This imbalance of -250 Gt/yr is within the range of -500 to 400 Gt/yr resulting from observations of grounded ice but is higher than the -60 ± 76 Gt/yr, mainly from West Antarctica, obtained from five years of satellite radar altimeter data (Wingham *et al.*, 1998). This discrepancy could arise from the adopted Pleistocene forcing model ICE-3G or from the underestimation, in the treatment of radar altimeter data, of the accumulation rate that could have hidden an imbalance larger than the -60 Gt/yr (Wingham *et al.*, 1998). When Greenland is added, right column, the tendency to higher lower mantle viscosities is reinforced. $\dot{J}_2$ portrays a single minimum at 22.0-22.8, in agreement with the now better defined minima of $\dot{J}_4$ and $\dot{J}_6$. $\dot{J}_3$ now splits into two possible viscosity solutions, with the highest viscosity still coherent with the patterns of the other zonals. χ^2 portrays a weak dependence on the upper mantle viscosity for all the zonals. Cheng *et al.* (1997) provide a clear indication for ice loss in Antarctica and for a further viscosity increase in the lower mantle once ice loss in Greenland is added to that in Antarctica. Intermediate values of ice loss in Greenland between 0 and -144 Gt/yr would provide lower mantle viscosities between 21.6-22.0 and 22.0-22.8. The χ^2 patterns relative to the Devoti *et al.* (2001) solution, Figure 5.7, are in close agreement with Cheng *et al.* (1997) as far as $\dot{J}_2$ and $\dot{J}_6$ are concerned, with higher resolution in the location of the minima, but differ in $\dot{J}_4$, portraying two possible viscosity solutions as expected, on the other hand, on the basis of Figure 5.3. $\dot{J}_{odd}$ cannot be directly compared with the separate

$\dot{J}_3$ and $\dot{J}_5$ in Cheng *et al.* (1997), being a summation of a larger number of odd zonals, but it is remarkable that it portrays some similarities with the Cheng *et al.* (1997) odd components. The lowest χ^2 value region in fact overlaps in the two SLR solutions, with a better defined minimum in Cheng *et al.* (1997) of $\chi^2 = 0.0 - 0.2$ with respect to 1.4-1.6 in Devoti *et al.* (2001), contaminated by several odd zonals. When ice loss in Greenland is added in the right column, the tendency towards higher lower mantle viscosities also appears in Figure 5.7. The lowest χ^2 value for $\dot{J}_{odd}$ and upper mantle viscosities of the order of 10^{20} Pa s now overlaps with the minima of the other zonals, improving the coherence with Cheng *et al.* (1997).

The $\dot{J}_l$ results can be strengthened by TPW simulations. Historical and modern polar motion data, as given in McCarthy and Luzum (1996), are now compared with modeled TPW predictions to validate, by means of an independent source, the results obtained solely on the basis of the SLR analysis. Figure 5.8 shows the TPW χ^2 patterns when the observational datum of 0.333 ± 0.008 arcsec/century is considered (McCarthy and Luzum, 1996) with the top and bottom panels corresponding to the same forcing of the left and right columns in Figures 5.4 and 5.5, respectively. Note the overlapping of the TPW and $\dot{J}_2$ χ^2 minima for a lower mantle viscosity of 21.8 when ice loss is limited to Antarctica, top panel. This finding solves the apparent paradox noted in the past consisting of TPW data systematically requiring higher values of lower mantle viscosities with respect to $\dot{J}_2$ when Pleistocene deglaciation is the only forcing mechanism (Vermeersen *et al.*, 1997). In the bottom panel, the TPW pattern favors an increase of lower mantle viscosity to 22.4 - 22.6, in agreement with $\dot{J}_l$ simulations for melting in Greenland of -144 Gt/yr. Ice mass imbalance in Greenland does not degrade the coherence between the lower mantle viscosity inferred from the $\dot{J}_l$ and TPW obtained in the top panel.

We have thus obtained that the χ^2 viscosity patterns based on the low degree $\dot{J}_l$ in Cheng *et al.* (1997) are the same if the rate of melting in Antarctica is constrained to -250 Gt/yr, corresponding to a sea-level rise of 0.7 mm/yr, and if the lower mantle is more viscous than the upper mantle. This scenario is coherent with Devoti *et al.* (2001), even if the $\dot{J}_3$ and $\dot{J}_5$ zonals are contaminated by higher degree contributions. The tendency of the lower mantle viscosity solutions towards values higher than 10^{21} Pa s when melting in Antarctica is added to Pleistocene deglaciation agrees with previous indications based on the temporal variations of the gravity field (Mitrović and Peltier, 1993; Nerem and Klosko, 1996). In particular, our inference of lower mantle viscosity when Antarctica is the major contributor to present mass imbalance is in close agreement with the findings of a previous analysis by Johnston and Lambeck (1999) based solely on $\dot{J}_2$, on the assumption that the non-steric component of sea-level rise is 1 mm/yr. The novelty of our analysis stands on the release of this assumption on sea-level rise thanks to the joint use of the accurately detected even and odd

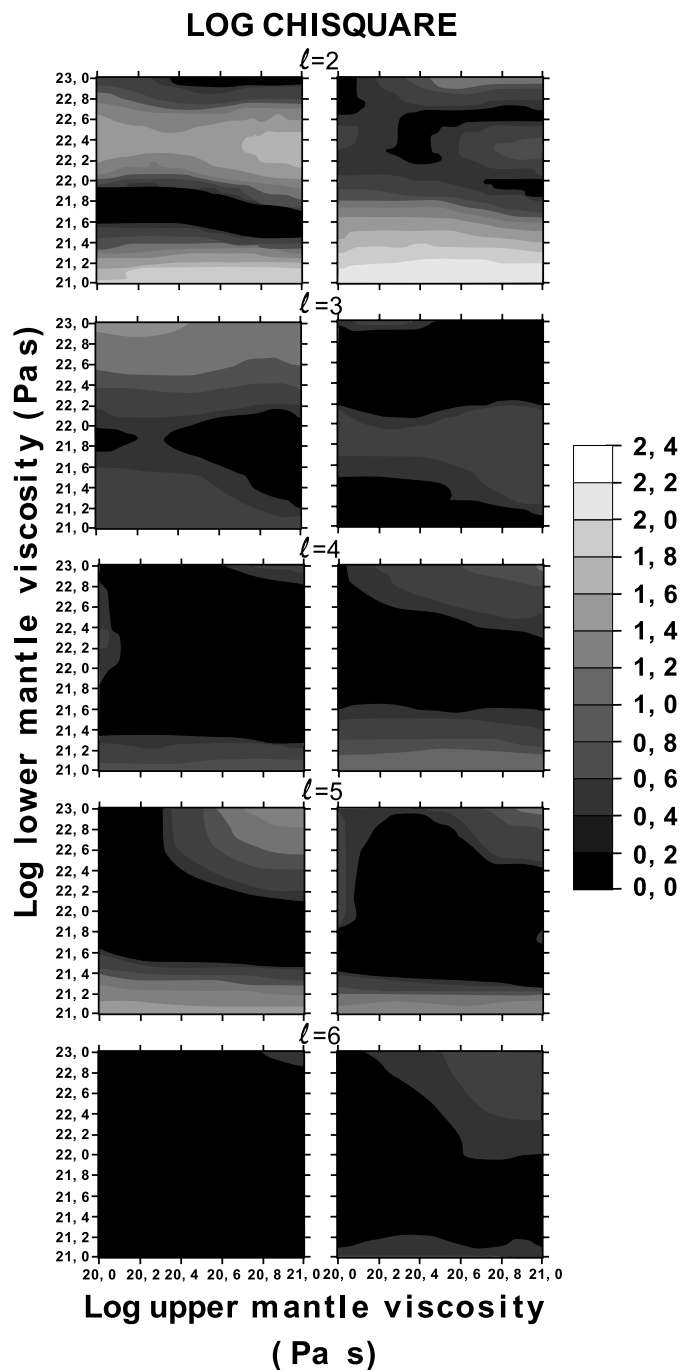


Figure 5.6. χ^2 patterns for the $\dot{J}_l$ equation (5.1), as a function of upper and lower mantle viscosities, these parameters varying as shown by the horizontal and vertical scales. Observational data are taken from the Cheng *et al.* (1997) solution, Tables 5.1 and 5.2. The minima correspond to the darkest grey. Pleistocene plus ice loss in Antarctica (-250 Gt/yr) are considered in the left column. In the right column ice loss of -144 Gt/yr in Greenland is added.

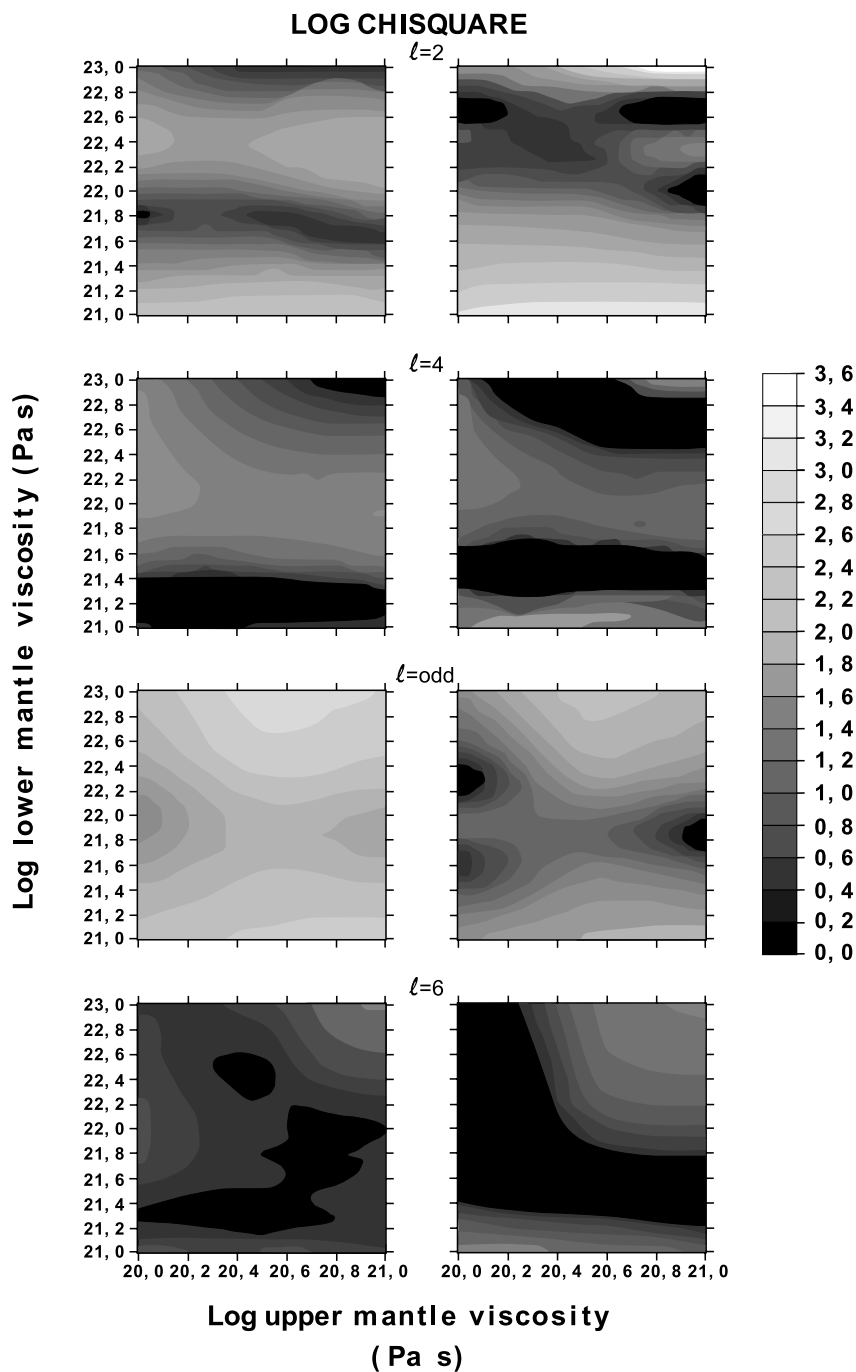


Figure 5.7. χ^2 patterns for the $\dot{J}_l$, as in Figure 5.6, for the Devoti *et al.* (2001) solution, with $\dot{J}_{\text{odd}}$ given by the combination of equation (5.2). The left and right columns correspond to the same forcing in Figure 5.6.

zonal components of the SLR-retrieved gravity field. The non-steric component of sea-level rise that results from our study is at most 1.1 mm/yr, of which 0.7 from Antarctica and 0.4 mm/yr from Greenland, within the best estimate total sea level rise of 1.8 mm/yr, which includes 0.4 mm/yr of thermal expansion of the oceans. Our analysis reconciles lower mantle viscosity inferences from Pleistocene deglaciation with those based on the long-wavelength, static components of the geoid anomalies and true polar wander driven by internal mass heterogeneities, which indicate a lower mantle more viscous than the upper mantle (e.g., Hager, 1984; Sabadini and Yuen, 1989; Ricard *et al.*, 1993; Forte and Mitrova, 1996). This unification of the viscosity inferences resulting from geodynamical processes spanning different time scales, from thousands to hundreds of millions of years, is made possible nowadays by means of the accurate detection of the even and odd zonal components of the time-dependent gravity field, proving that the SLR technique will remain a fundamental tool to constrain the global dynamics of our planet. Dedicated satellite gravity missions like GRACE are foreseen to contribute substantial improvements towards the higher harmonics, although it remains to be seen whether these relatively short-lived space missions will be able to detect the secular signals (e.g., Kaufmann, 2000; Velicogna and Wahr, 2002; Wahr and Davis, 2002).

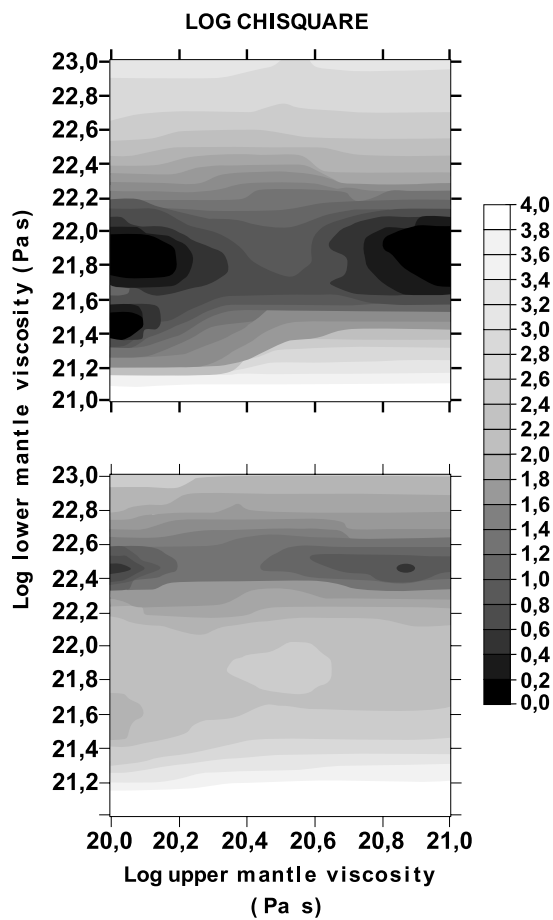


Figure 5.8. χ^2 patterns for TPW. The top panel corresponds to Pleistocene deglaciation and ice loss in Antarctica, while ice loss in Greenland is added in the bottom panel.

Chapter 6

SEA-LEVEL CHANGES

1. INTRODUCTION

To the general public it seems a rather obvious fact that if the ice sheets of Antarctica and Greenland melted, sea level would rise. In fact, there is wide-spread concern that global warming of the Earth might induce melt of the present-day large ice sheets and thereby induce a global sea-level rise, leading to floodings of lowlands. Although the link between ice melt and sea-level rise might seem obvious, the relation between ice mass changes and sea-level variations is more complicated. It might sound strange when one first hears about it, but ice melt could actually induce a sea-level drop at some places on the Earth's surface. And perhaps it is even more surprising that this has been known for more than a century.

In this oversimplified picture of ice melt and sea-level rise we neglect that ice has mass. Mass has the property of attracting other masses around it. So an ice sheet on a continent attracts the water of the oceans. Thus, the water near an ice sheet will be elevated with respect to a situation in which this gravitational interaction does not occur. If the ice sheet melts, this gravitational attraction effect, or self-gravitation, disappears too. Woodward (1888) showed that it is not a small, negligible effect. He derived the following formula for the ratio of the change in sea level with self-gravitation taken into account and the sea-level change without the self-gravitation of the ice sheet taken into account. If the ice sheet, modeled as a point source, were to melt, this ratio is:

$$\left(\frac{1}{2 \sin(\theta/2)} - 1 - \frac{\rho_E}{3\rho_w} \right) / \frac{\rho_E}{3\rho_w}. \quad (6.1)$$

In equation (6.1) θ is the angular distance from the ice sheet, ρ_E the mean density of the Earth, and ρ_w the density of the water. Equation (6.1) shows that the ratio is not dependent on the total mass of the ice sheet. A derivation

of formula (6.1) can also be found in Farrell and Clark (1976). From equation (6.1) one can easily compute that sea level will drop in the oceans as a result of ice melt within a distance of about 20 degrees (on the Earth's surface equal to about 2200 km) from the former ice sheet. Even within 60 degrees (equal to about 6700 km), sea level will rise less than if the amount of water from the ice sheet had been distributed uniformly ('eustatically') over the oceans. But, on the other hand, at distances exceeding 60 degrees sea level will rise more than the eustatic value. So if all the Greenland ice were concentrated on the southern tip of Greenland and were to melt, then the effect of self-gravitation would cause a sea-level drop at the coast of Iceland, a smaller sea-level rise than the eustatic one at the coast near New York and a larger sea-level rise than the eustatic one at the coast of Australia.

The relation between ice melt and sea-level change is clearly not so simple as we generally might think. Although the total amount of ice that melts is equal to the total amount of water that is added to the oceans, the redistribution of the melt water is not uniform. The situation becomes even more complicated if one considers the fact that the solid earth is not rigid. Placing a load on the Earth's surface will cause an immediate (elastic) subsidence of the surface underneath the load, followed by a time-dependent and spatially dependent subsidence and uplift pattern caused by viscoelastic flow in the Earth's interior, as we have learned from Chapters 1 and 2. This implies that if ice and water are redistributed over the Earth's surface, the solid Earth will react to it. Solid-Earth deformation due to a changed surface load can still be observed today in Canada and Scandinavia. Some 10,000 years ago the last of the great Ice Ages ended with the melt-down of the Laurentide and Fennoscandian ice sheet complexes, which covered these regions with a maximum height of about 3 km. Today the Earth is still rebounding to the disappearance of the Pleistocene ice-sheets, with maximum uplift rates of about 1 cm per year near the center of the Bothnic Gulf and near the south of Hudson Bay. So the relation ice melt = sea-level rise, which we have had to modify because of the effect of self-gravitation, must be expanded to a three-component relation (continental) ice / sea level / solid Earth. Ice melt will cause a sea-level change and both components will induce solid-Earth deformation. But solid-Earth deformation in turn will induce a sea-level change again. Whereupon this sea-level change will induce solid-Earth deformation, and so on. It is clear that this relationship is a non-linear one. The formulations for this relationship were derived in the 1970s and have become known as the 'sea-level equation', which is an integral equation. In a very condensed form the equation for the relative sea-level change reads (Farrell and Clark, 1976)

$$S = \rho_I \frac{\phi}{g} * L + \rho_w \frac{\phi}{g} * S + C, \quad (6.2)$$

in which S denotes the change in sea level, L the change in (continental) ice thickness, ϕ the Green function for the variation in the gravitational potential, ρ_I the density of the ice and g surface gravity. C is a constant which is to be determined by invoking the condition that the total amount of ice change is equal to the total amount of sea-level change. The asterisk denotes time convolution. Note that the sea-level change S is both on the left-hand side and on the right-hand side of equation (6.2).

With this sea-level equation, by which the sea-level change and solid-Earth deformation interrelationship as function of ice mass variations (and solid-earth models) can be handled, present-day sea-level variations can be modeled adequately, both for sea-level changes due to Pleistocene deglaciation and to recent continental ice mass changes (e.g., Farrell and Clark, 1976; Peltier and Andrews, 1976; Wu and Peltier, 1983; Nakada and Lambeck, 1987; Lambeck *et al.*, 1990; Mitrović and Peltier, 1991; Johnston, 1993; Di Donato *et al.*, 2000; Mitrović and Vermeersen, 2002).

Melting ice causes a redistribution of mass of the rotating Earth. A redistribution of mass affects the moments and products of inertia. A deformable rotating body will react to such changes by adjusting rotation rate and position of its rotation axis with respect to the body's surface topography. For the Earth it implies that the three components ice-mass changes / sea level variations / solid-Earth deformation will affect the Earth's rotation. But changes in the Earth's rotation in turn could effect the three components again. Sea level and solid-Earth deformation can be affected by the change in centrifugal force accompanying rotation changes. Ice can be affected as a redistribution of mass of a rotating body will generally affect the position of the rotation axis with respect to the body's topography (not with respect to its surroundings: to an outside observer it looks as if the body is shifting underneath its rotation axis, while the rotation axis remains fixed with respect to the inertial system of its surroundings). For the Earth this implies that the distribution of oceans and (partly ice-covered) land over the planet's surface will change with respect to the plane of the Earth's orbit around the sun, although the angle of the rotation axis with respect to this plane remains fixed. Such a global shift can induce changes in the ocean's currents, enhance or reduce the amount of continental areas around the poles available for ice to settle on, and increase or decrease ice melt due to changes in insolation angle. It is clear that in order to be self-consistent we have to extend the three-component relationship ice / sea level / solid-Earth deformation with a fourth component: polar wander.

2. SEA-LEVEL VARIATIONS, GEOID AND GRAVITY ANOMALIES DUE TO PLEISTOCENE DEGLACIATION

The importance of Pleistocene deglaciation on the interpretation of sea-level variabilities has been addressed by several authors. For example, Lambeck and Nakiboglu (1984) find that post-glacial rebound may contribute between 30 and 50 per cent of the present-day secular rise in sea level of 1.8 mm/yr (Douglas, 1995). Although there are many uncertainties in estimating such a percentage, such figures show the potential importance of including solid-Earth deformation processes in studies on sea-level variations.

Both the absolute value of the contemporary global (eustatic) sea-level rise and the effect that post-glacial rebound has on contemporary sea-level variations are not well known at the moment. Whereas the former is mainly a matter of measurements, the uncertainty in the contribution that post-glacial rebound has on present-day global sea-level variations is mainly due to insufficient knowledge about the Pleistocene ice sheets and the rheological properties of the Earth. However, differences in modeling approaches for establishing the relationship between post-glacial rebound and sea-level variations can also be a source of discrepancies.

The results shown in this chapter make use, for the first time in glacial-rebound induced sea-level studies, of an analytical scheme described in previous chapters in dealing with stratified earth models.

2.1 MATHEMATICAL FORMULATION

In the present analysis we show some results mainly focussed on the sensitivity of density and viscosity stratification in sea-level modeling.

The sea-level variations and geoid are based on the following equations, in relation to the Green functions Φ and Γ for the gravitational potential perturbation and radial displacement of the solid surface.

The geoid anomaly G and the vertical displacement R are given by Mitrovica and Peltier (1989), whose expressions are reproduced to adapt our formalism to their sea-level change theory

$$G(\vartheta, \varphi, t) = \frac{1}{g} \int_{-\infty}^t \iint_{\Omega} a^2 \sigma(\vartheta', \varphi', t') \Phi(\gamma, t - t') d\Omega' dt' \quad (6.3)$$

$$R(\vartheta, \varphi, t) = \int_{-\infty}^t \iint_{\Omega} a^2 \sigma(\vartheta', \varphi', t') \Gamma(\gamma, t - t') d\Omega' dt' \quad (6.4)$$

where a is the mean radius of the Earth, g is the surface gravitational acceleration, γ represents the angular distance between the impulse load point (ϑ', φ')

and the observation point (say ϑ, φ), t is the time, Ω represents the complete solid angle and σ is the surface load whose spherical harmonic coefficients are

$$\sigma_{lm}(t) = \sum_{n=1}^N \delta\sigma_{lm}^n H(t - t_n), \quad (6.5)$$

with N denoting the number of time steps in which the surface load is subject to a change; H is the Heaviside step function and

$$\delta\sigma_{lm}^n = \rho_I \delta I_{lm}^n + \rho_W \delta S_{lm}^{n,e}, \quad (6.6)$$

where ρ_I and ρ_W are, respectively, the densities of ice and water, δI_{lm}^n and $\delta S_{lm}^{n,e}$ denote the spherical harmonic coefficients of the increments in the ice masses and eustatic sea level at the various time steps

$$\delta S_{lm}^{n,e} = \frac{\rho_I}{\rho_W} \frac{4\pi a^2}{A_o} \delta I_{00}^n C_{lm}, \quad (6.7)$$

where C_{lm} are the spherical harmonic coefficients of the ocean function, defined to be unity over the ocean and zero over the remaining surface of the Earth, and A_o is the present area of the oceans; equation (6.7) is equation (53) in Mitrovica and Peltier (1991).

The Green functions Φ, Γ are given by

$$\Phi(\gamma, t) = \frac{ag}{M_E} \sum_{l=0}^{\infty} \left\{ (1 + k_{le}^L) \delta(t) + \sum_{j=1}^M k_{lj}^L \exp(s_{lj}t) \right\} P_l(\cos \gamma), \quad (6.8)$$

$$\Gamma(\gamma, t) = \frac{a}{M_E} \sum_{l=0}^{\infty} \left\{ h_{le}^L \delta(t) + \sum_{j=1}^M h_{lj}^L \exp(s_{lj}t) \right\} P_l(\cos \gamma) \quad (6.9)$$

which are equations (8) by Mitrovica and Peltier (1991), except equation (6.8) includes the direct effect of the load, namely the term 1 in the first brackets; P_l is the Legendre polynomial given by equation (1.34), M_E is the mass of the Earth, δ is the Dirac delta function and k_{le}^L, h_{le}^L denote the spherical harmonic components of the elastic load Love numbers for the perturbation to the gravitational potential and the radial displacement, respectively. The terms k_{lj}^L, h_{lj}^L represent the viscous components of the same numbers for the individual modes j with negative inverse relaxation times s_{lj} .

Solving the integration in time and over the surface of the Earth in equations (6.3) and (6.4) yields

$$G(\vartheta, \varphi, t) = G^E(\vartheta, \varphi, t) + G^V(\vartheta, \varphi, t) \quad (6.10)$$

$$R(\vartheta, \varphi, t) = R^E(\vartheta, \varphi, t) + R^V(\vartheta, \varphi, t) \quad (6.11)$$

where the superscripts E and V denote the elastic and viscous components, as given below (Di Donato *et al.*, 2000)

$$G^E(\vartheta, \varphi, t) = \frac{4\pi a^3}{M_E} \sum_{l=2}^{\infty} \frac{(1 + k_{le}^L)}{(2l + 1)} \sum_{m=-l}^l \sigma_{lm}(t) Y_{lm}(\vartheta, \varphi) \quad (6.12)$$

$$R^E(\vartheta, \varphi, t) = \frac{4\pi a^3}{M_E} \sum_{l=2}^{\infty} \frac{h_{le}^L}{(2l + 1)} \sum_{m=-l}^l \sigma_{lm}(t) Y_{lm}(\vartheta, \varphi) \quad (6.13)$$

$$G^V(\vartheta, \varphi, t) = \frac{4\pi a^3}{M_E} \sum_{l,m} \left\{ \sum_{j=1}^M \frac{k_{lj}^L}{(-s_{lj})(2l + 1)} \sum_{n=1}^N \delta \sigma_{lm}^n [1 - \exp(s_{lj}(t - t_n))] \right\} Y_{lm}(\vartheta, \varphi) \quad (6.14)$$

$$R^V(\vartheta, \varphi, t) = \frac{4\pi a^3}{M_E} \sum_{l,m} \left\{ \sum_{j=1}^M \frac{h_{lj}^L}{(-s_{lj})(2l + 1)} \sum_{n=1}^N \delta \sigma_{lm}^n [1 - \exp(s_{lj}(t - t_n))] \right\} Y_{lm}(\vartheta, \varphi), \quad (6.15)$$

with Y_{lm} given in equation (1.28).

The relative sea-level change, which will be denoted by RSL from now on instead of S as in Mitrovica and Peltier (1989), is the difference between the changes in the geoid and radial displacement projected over the surface of the oceans (Farrell and Clark, 1976), namely

$$RSL(\vartheta, \varphi, t) = C(\vartheta, \varphi) (G(\vartheta, \varphi, t) - R(\vartheta, \varphi, t)) \quad (6.16)$$

The sea-level change has the following explicit expression (equation (18) in Di Donato *et al.*, 2000)

$$RSL(\vartheta, \varphi, t) = C(\vartheta, \varphi) \sum_{l,m} \left\{ E_l T_l \sigma_{lm}(t) + T_l \sum_{n=1}^N \delta \sigma_{lm}^n \beta_l(t - t_n) \right\} Y_{lm}(\vartheta, \varphi), \quad (6.17)$$

with

$$E_l = 1 + k_l^E - h_l^E \quad (6.18)$$

$$\beta_l(t - t_n) = \sum_{j=1}^M \frac{(k_{lj}^L - h_{lj}^L)}{(-s_{lj})} [1 - \exp(s_{lj}(t - t_n))], \quad (6.19)$$

$$T_l = \frac{4\pi a^3}{M_E(2l + 1)}. \quad (6.20)$$

Equations (6.18), (6.19) and (6.20) are equations (16) and (22) in Mitrovica and Peltier (1991).

The self-consistent sea-level calculation requires the solution of an integro-differential equation that can be solved by means of the pseudo-spectral approach first outlined and implemented by Mitrovica and Peltier (1991), where a detailed and clear discussion of the method can be found. The sea-level rates due to glacial isostatic adjustment that we present here have been calculated by means of this formalism, while the geoid computations are based on the so-called eustatic approximation in which the water is distributed uniformly in the oceans (Wu and Peltier, 1983). In order to make estimates of the geoid and free-air gravity anomalies due to post-glacial rebound, the eustatic approximation is certainly satisfactory as these global signatures are relatively insensitive to the detailed history of water redistribution in the oceans.

2.2 SEA-LEVEL VARIATIONS, THE GEOID AND FREE-AIR GRAVITY ANOMALIES

In the computations that follow, 5- and 31-layer earth models are considered, whose parameters will be those in Table 2.2 and Table 4.1. The lithospheric thickness is fixed at 120 km and the upper mantle viscosity at 10^{21} Pa s, while the viscosity of the lower mantle will be 10^{21} Pa s for model 1, 5×10^{21} Pa s for model 2 and 10^{22} Pa s for model 3. Models 1 and 3 represent two end-members for a homogeneous and strongly stratified mantle, while model 2 is an intermediate case. Model 3 is in closer agreement with mantle convection results (Ricard and Vigny, 1989) and with the $\dot{J}_l$ results from the SLR analyses of Chapter 5.

For the loading history, the ICE-3G model by Tushingham and Peltier (1991) shown in Figure 4.3 is adopted. In the calculations of the geoid and gravity anomalies summation is carried up to degree 80. This is a truncation high enough to estimate the major signal and its pattern.

Figure 6.1 provides the present-day geoid anomaly due to Pleistocene deglaciation for the 5-layer earth model 1. The peak anomalies of - 9 m, - 4 m and - 10 m are located over Hudson Bay, the Gulf of Bothnia and the Antarctic region,

respectively, where the largest ice sheets were located. As already observed by Mitrovica and Peltier (1989) for the northern regions, these anomalies are comparable with the observed geoid in the range of harmonic degrees $10 < l < 22$. Longer wavelengths are in fact correlated with geoid anomalies due to mantle convection (Mitrovica and Peltier, 1989). The pattern of geoid anomalies is rather smooth, in agreement with the dominant low frequency content of geopotential perturbation. This issue will be considered later on when dealing with gravity anomalies and their richer content at high harmonic degrees. A remarkable feature of this geoid is the positive anomaly of about 1 m located in the Atlantic and Pacific oceans, clearly due to the mantle material which has flowed away from the glaciated regions during the phase of glaciation towards the equatorial regions. In these oceanic regions where the geoid anomaly is positive, the present-day rates of change of the geoid predicted by Mitrovica and Peltier (1991) using the same ice model and mantle viscosity profile are negative, indicating that this geoid anomaly is decreasing to restore isostatic equilibrium.

Figure 6.2 provides the geoid anomaly for the 5-layer earth model 3. With respect to Figure 6.1, it is remarkable that the geoid pattern is maintained, except for the peak values of the anomalies, which are larger than in the homogeneous model. The highest anomaly is of course caused by the larger amount of isostatic disequilibrium maintained by the high viscosity of model 3, in comparison with the homogeneous one. The peak values of the anomalies are now -36, -12 and -30 m, about a factor four larger than for model 1. It is also remarkable that even the positive anomalies in the oceans are larger than the corresponding ones in Figure 6.1, in agreement with the larger amount of mantle material stored beneath the ocean basins that, due to the higher viscosity, has not flowed yet under the continental areas to fill the large mass deficit.

In Figure 6.3, the free-air gravity anomaly is provided for model 1. The peak values of the gravity anomalies are located, as expected, in the center of the deglaciated areas and are, respectively, -18 mgal over the Antarctic, -10 mgal over Hudson Bay and -6 mgal over the Northern part of Europe in agreement with the findings first reported by Mitrovica and Peltier (1989). The pattern of the gravity anomalies has a close resemblance to the geoid. As expected, the pattern of the free-air gravity anomalies is not as smooth as the pattern of the geoid, due to the faster decay with distance of the signal from the anomalous masses, indicating a richer high-frequency content in the free-air gravity anomalies in comparison to the geoid ones.

In order to appreciate the high-frequency content in the free-air gravity anomaly, the case in which only spherical harmonic coefficients in the degree range $l = 80, 200$ have been retained is considered in Figure 6.4. A signal of -3 mgal is obtained in the Antarctic region and -1 mgal in the other deglaciated centers.

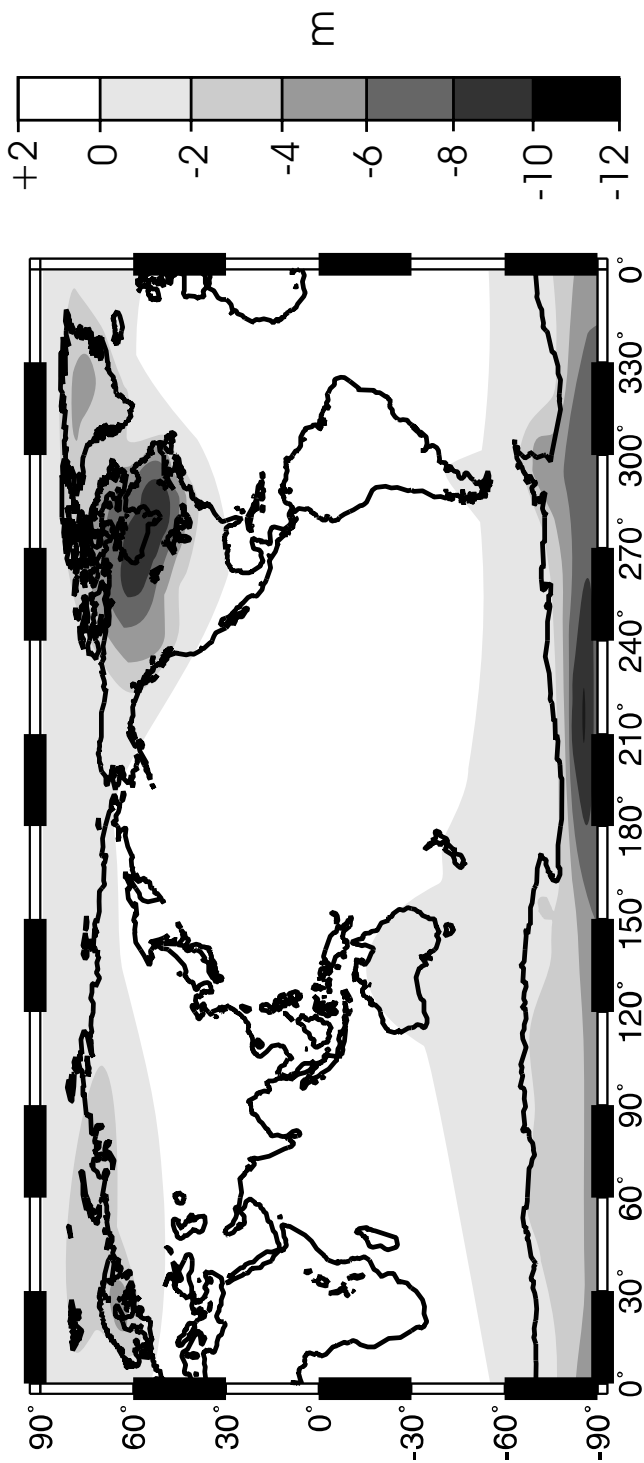


Figure 6.1. Global map of the present-day geoid anomaly due to Pleistocene deglaciation for earth model 1 ($l = 2, 80$).

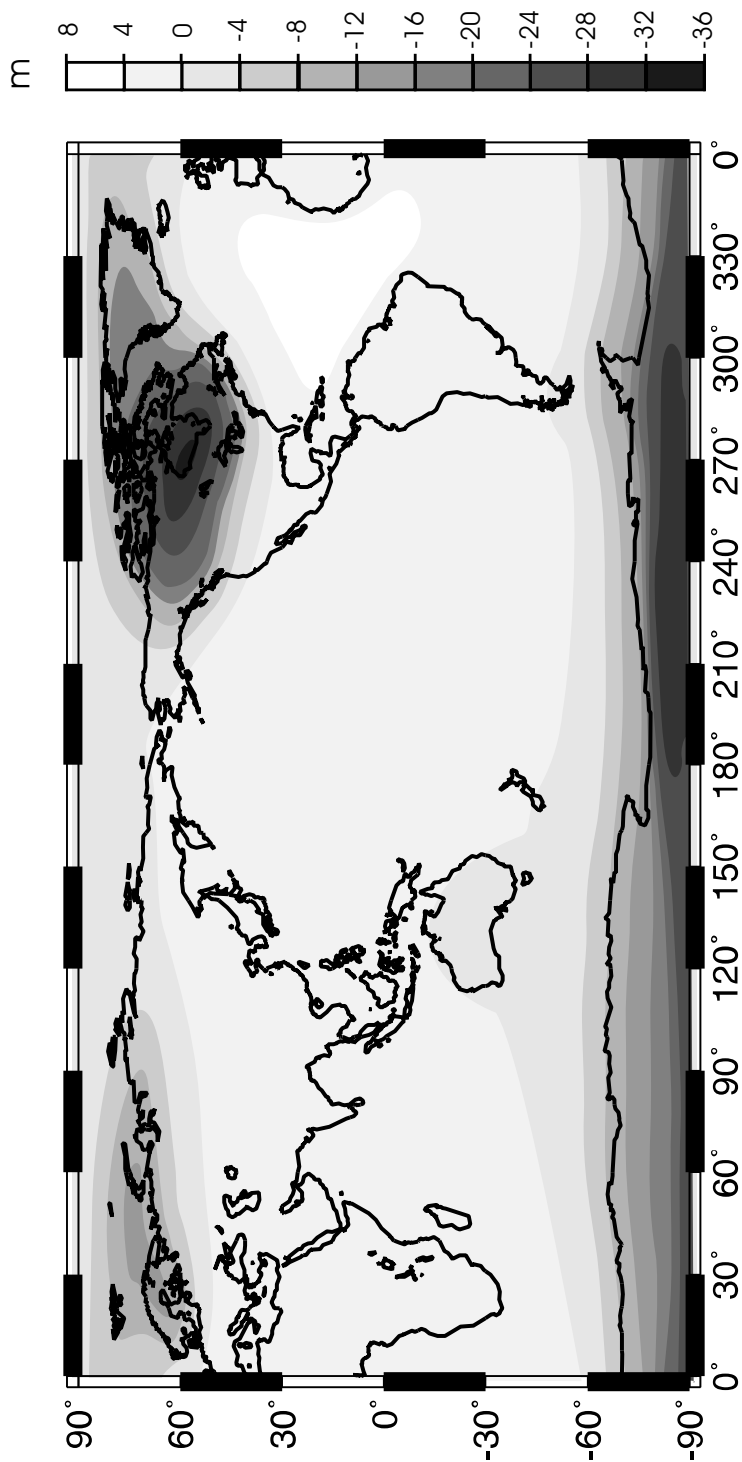


Figure 6.2. Global map of the present-day geoid anomaly due to Pleistocene deglaciation for earth model 3 ($l = 2, 80$).

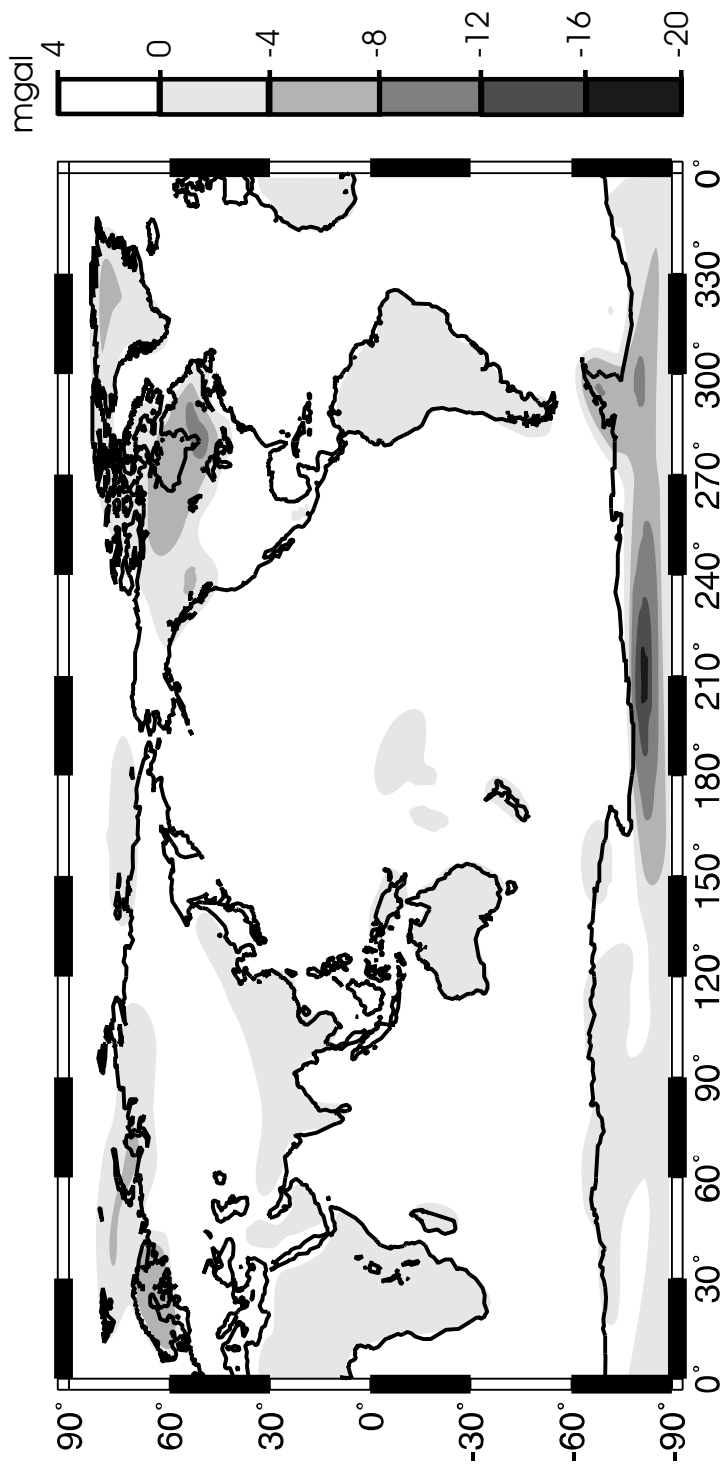


Figure 6.3. Global map of the present-day free-air gravity anomaly computed using earth model 1 ($l = 2, 80$).

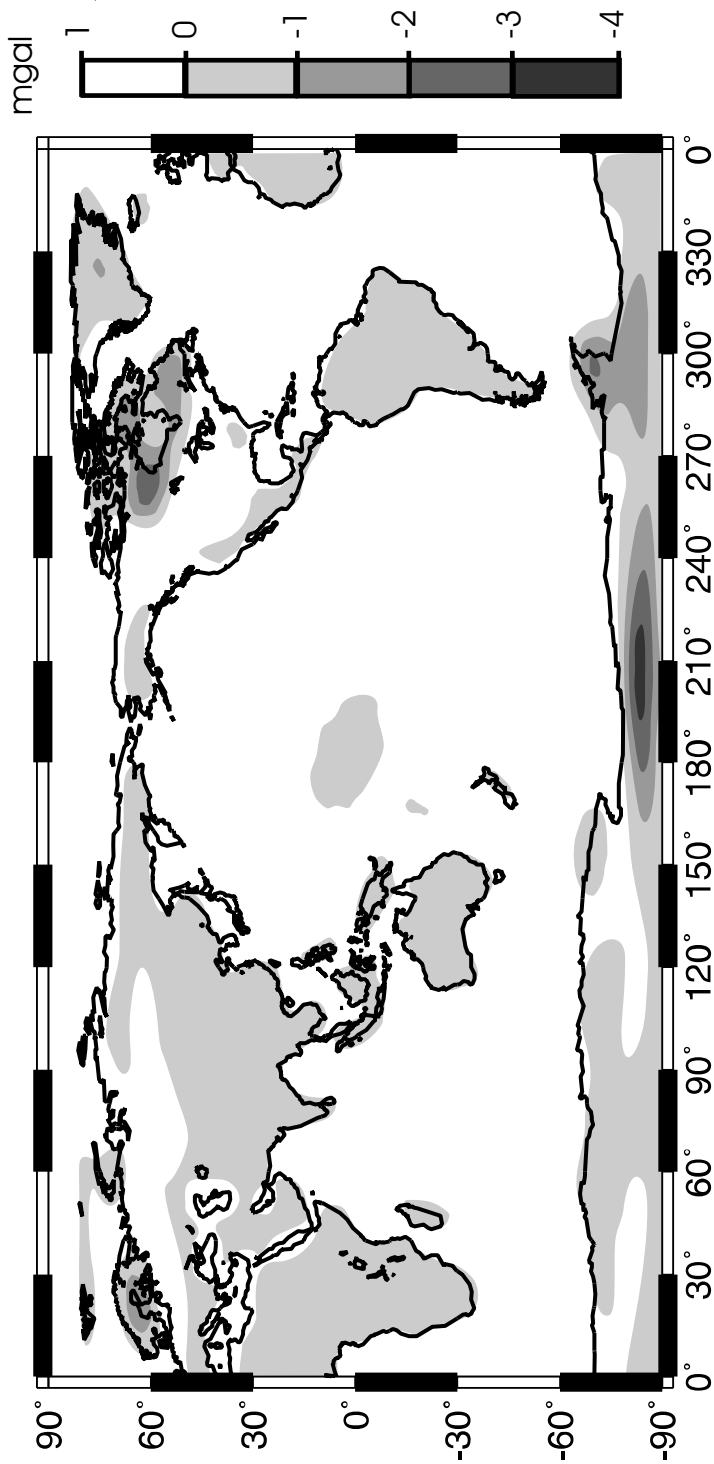


Figure 6.4. Global map of the present-day free-air gravity anomaly computed for earth model 1 by retaining only the spherical harmonic coefficients in the degree range $l = 80, 200$.

Figures 6.5a and 6.5b show the predicted present-day rates of sea-level change for the 31-layer earth model 2, in northern Europe and in the Mediterranean region, respectively. The lower mantle viscosity of model 2 is consistent with inferences based on post-glacial relative sea-level variations in northern Europe (Lambeck *et al.*, 1990) and numerical predictions of post-glacial sea-level change in southern Europe (Mitrovica and Davis, 1995). The sea-level equation (6.16) is solved using the pseudo-spectral algorithm by Mitrovica and Peltier (1991) with a truncation at degree and order 256, so the spatial resolution is sufficient to model sea-level increase in small regions. The highest rates are obtained of course in the center of deglaciation, the Gulf of Bothnia. The uplift of the land causes a sea-level fall of -11 mm/yr there. On the periphery of the uplifting region, the land is subsiding due to the collapse of the peripheral bulge, causing a sea-level increase that, along the coastal areas of northern Europe, can be as high as 1 - 2 mm/yr. Sea-level increase along the coasts of northern continental Europe is subject to large variabilities, from 0.4 mm/yr near the French coast to 1 - 1.5 mm/yr along the coasts of The Netherlands and Germany. It should be noted that the collapse of the peripheral bulge decreases in central Europe. Further to the south, sea level is actually increasing, with rates in the order of 0.6 - 0.8 mm/yr in the central Mediterranean due to the subsidence of the sea bottom caused by the water load. The periphery of the Mediterranean Sea is characterized by a weak sea-level rise with rates from 0.2 to 0.4 mm/yr in the Adriatic Sea and about 0.4 mm/yr along the Mediterranean coast of France. Here the rates of sea-level change have small values due to the levering effect (e.g., Nakada and Lambeck, 1989): the subsidence of sea basins is contoured by the uplift of the surrounding continents. This sea-basin subsidence is due to mantle material presently flowing from the Mediterranean region towards the Fennoscandian region in northern Europe. These last values are comparable with the sea-level changes due to active tectonics in the central Mediterranean: finite-element calculations of active tectonics in peninsular Italy have indicated that overthrusting of the Apennines onto the Adriatic plate is responsible for a sea-level increase of 0.4 mm/yr (Negredo *et al.*, 1997). The peak value in the central Mediterranean due to post-glacial rebound is thus comparable with that due to active tectonics (Di Donato *et al.*, 1999).

The model results indicate that glacial isostatic readjustment of the solid Earth can induce sea-level variations in sea basins located in the far field with respect to the centers of deglaciated regions in the order of 1 mm/yr, of the same order of magnitude as those induced by tectonic processes in the Mediterranean, as shown in the following section.

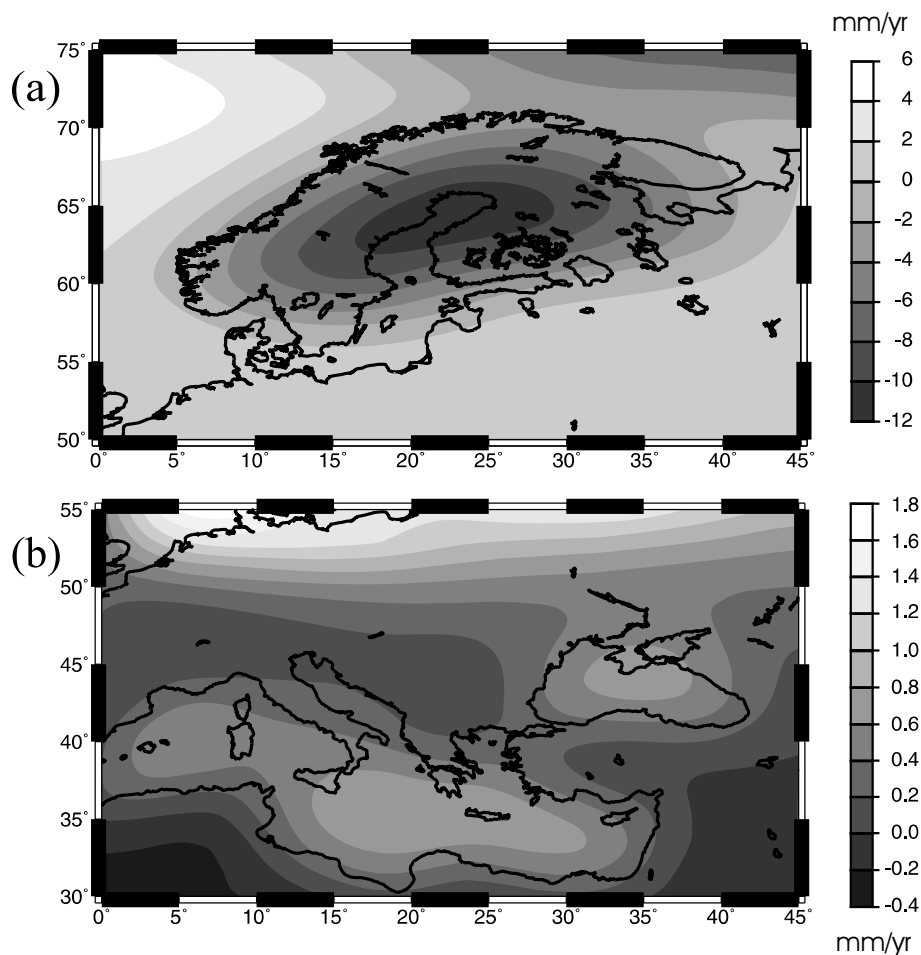


Figure 6.5. The present-day rates of sea-level change (a) in northern Europe and (b) in the Mediterranean region due to Pleistocene deglaciation for earth model 2 ($l = 2, 256$).

3. GLACIAL ISOSTATIC ADJUSTMENT (GIA) VS. TECTONIC PROCESSES: THE EXAMPLE OF THE MEDITERRANEAN SEA

The Adriatic Sea and the coastlines of peninsular Italy represent an important area for studying sea-level changes because model predictions can be used to gauge the risk of sea-level rise on the historically important cities of Venice and Ravenna. Furthermore, the current level of ancient Roman ruins in this part of the Mediterranean, relative to present-day sea level, provide high-quality sea-level records (Pirazzoli, 1998) (see Figure 6.6).

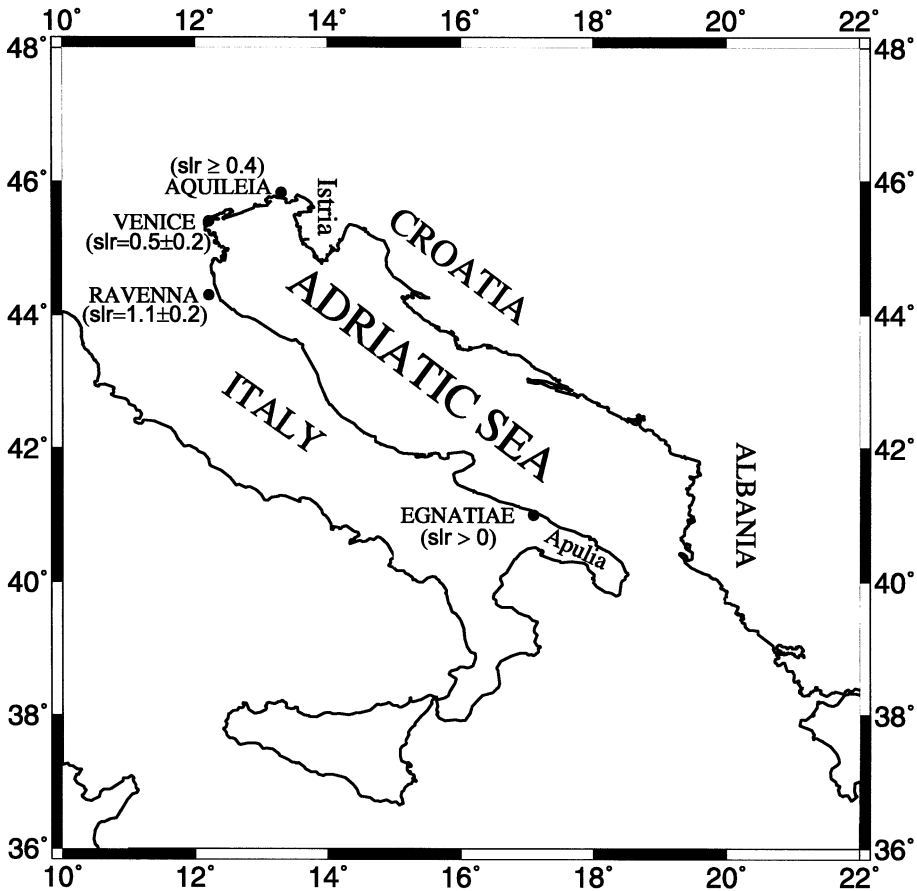


Figure 6.6. Map of the study area with site locations and observational sea-level rates in millimeters per year superimposed. Error bars for Venice and Ravenna are also included (Figure 1 in Di Donato *et al.*, 1999).

For example, archaeological evidence in the Istria promontory in the north Adriatic coast of Croatia indicates a sea-level rise of 0.7 mm/yr (Pirazzoli, 1998; Degrassi, 1955). At the Roman fluvial harbor of Aquileia in the northernmost part of the Adriatic, sea level has risen by at least 0.8 m since the first half of the first century A.D., providing a lower bound for the rate of sea-level rise of 0.4 mm/yr (Schmiedt, 1979). In Ravenna, Roman ruins are currently located 2 - 3 meters below sea level (Roncuzzi, 1970), suggesting a rate of sea-level rise in the range of 1.0 - 1.5 mm/yr, or 1.1 ± 0.2 mm/yr (Pirazzoli, 1998), while Flemming (1992) estimates a sea-level rise of 0.5 mm/yr in Venice, to which we can safely attribute conservative error bounds of ± 0.2 mm/yr following Flemming (1992). At Egnatiae, in Apulia, graves, harbor constructions and

a partially submerged Roman caldarium indicate a net sea-level rise in the southern part of the Adriatic coasts of Italy (Pirazzoli, 1998). These sea-level data are all based on archaeological ruins and are not affected by the subsidence induced by water extraction, which has hit the cities of Ravenna and Venice in the last 40 yr, or by neotectonic faulting, absent in the area.

These archaeological constraints are characterized by some potentially significant geographic variability, although they indicate a remarkably coherent sea-level rise in the northern Adriatic sea in the order of 1.0 mm/yr in the last 2,000 years. This change is unlikely to reflect a global signal, since the estimated present-day eustatic sea-level rise of 1.8 mm/yr has not persisted over the last two millennia (Douglas, 1995). It is reasonable, therefore, to look for other causes to explain these data, and the most obvious candidates are active tectonics and GIA. Although the impact of both active tectonics and GIA on sea-level changes is widely recognized (Mitrovica and Peltier, 1991; Pirazzoli, 1997), the tectonic contribution to sea-level data has to date been quantified only indirectly on the basis of residuals between sea-level observations and modeled GIA effects (Lambeck, 1995). In contrast to this indirect approach, in the following the effects of both these processes are self-consistently taken into account using tectonic and GIA models, as first shown in Di Donato *et al.* (1999). The resulting pattern of predicted sea-level changes can be compared directly with sea-level observations. This modeling is appropriate for time scales ranging between 10^3 and 10^5 yr and, thus, the results can be compared with the above archaeological data, which have a 10^3 yr time scale.

Active tectonic processes in the central Mediterranean, namely the Africa-Eurasia convergence and subduction in the southern Tyrrhenian, are modeled by means of finite-element solutions in a half-space domain. The model, which extends earlier work by Negredo *et al.* (1997), allows lateral variations in the rheology of the lithosphere and mantle which are consistent with tectonic structures in the central Mediterranean. Figure 6.7 is a cartoon representing the mesh of the finite-element model of peninsular Italy, which includes the deep tectonic structures of the central Mediterranean.

In contrast to that study (Negredo *et al.*, 1997), the present tectonic model is characterized by a realistic geometry of the Apennine chain overthrusting onto the Adriatic domain (Figure 6.7). The push of the African continent is indicated by the thick arrow directed roughly to the north, in agreement with a recent VLBI solution (Lanotte *et al.*, 1996). The Tyrrhenian subduction is portrayed by the deep lithospheric structure beneath the Calabrian Arc, with geometric characteristics in agreement with seismic tomography (e.g., Spakman, 1990). The tectonic structures, based on geological and seismological data, are sufficiently well constrained so as not to require a sensitivity analysis based on the parameters characterizing the tectonic model. The active push of Africa and the slab pull underneath the Calabrian Arc are responsible for deformation of

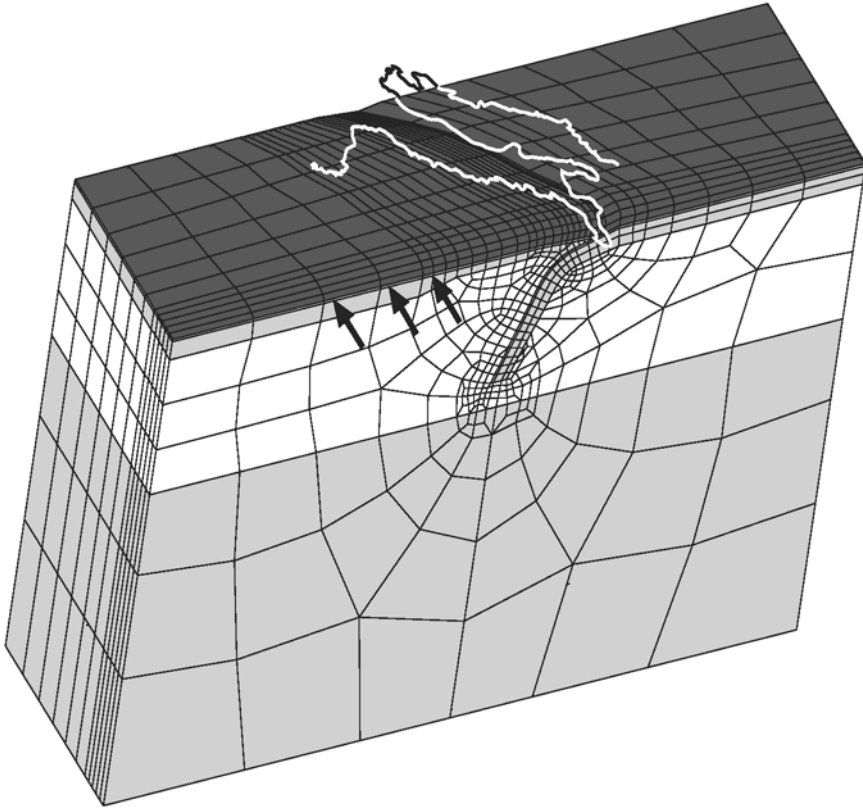


Figure 6.7. Cartoon portraying the mesh of the finite element tectonic model of the central Mediterranean. The lithosphere, upper and lower mantle are indicated. The dipping plate is the subducted oceanic lithosphere beneath the Calabrian Arc in the southern part of peninsular Italy. The coastlines of Italy are drawn on the top, with the thick curve with the finer mesh indicating the decoupling zone of the Apennines with respect to the Adriatic plate on the right. The horizontal arrows indicate the push of Africa; slab pull is activated in the subducted lithosphere. Redrawn from Figure 2 in Di Donato *et al.* (1999).

the whole domain situated between the African block and the Alps, in particular of the coastlines of peninsular Italy.

The redistribution of water due to ice-sheet disintegration is a global process. Predictions of sea-level variations due to GIA are based on the formulation provided at the beginning of this chapter. The predictions incorporate the effects of deglaciation, based on the ICE-3G reconstruction of Tushingham and Peltier (1991), on crustal and sea surface changes in the Mediterranean.

The glacially induced sea-level variations in the area arise from various coupled processes: loading associated with Pleistocene meltwater filling the Mediterranean, the rebound in Fennoscandia that drives mantle flow from the Mediterranean region to the uplifting area centered in the Gulf of Bothnia and long-wavelength motion of the water away from the Mediterranean and towards subsiding regions at the periphery of previously glaciated areas (Mitrovica and Davis, 1995), as shown in section 6.2.2. The tectonic and postglacial rebound models have consistent values for the thickness and rheology of the lithosphere and the viscosity within the upper and lower mantle. The elastic structure is constrained by the seismic model PREM (Dziewonski and Anderson, 1981) and the earth model has a 120 km-thick elastic lithosphere for modeling the GIA. Both the models are characterized by an upper mantle viscosity of 10^{21} Pa s and a viscosity increase in the lower mantle of a factor 30, in agreement with long-wavelength geoid modeling (Ricard and Vigny, 1989) and the results in Chapter 5. This viscosity profile reconciles the inference derived from long-term tectonic processes (Ricard and Vigny, 1989) with glacio-isostatic analyses that indicate substantial viscosity increase in the lower mantle (Mitrovica and Forte, 1997). Equations of section 6.2.1 and the iterative pseudo-spectral formalism described in Mitrovica and Peltier (1991) have been used in order to determine the gravitationally self-consistent sea-level changes described hereafter. The tectonic model, which is not self-gravitating, assumes that the geoid does not change during deformation. The sea-level change is, in this case, due to the motion of the sea floor. This assumption is justified by results from studies of sea-level changes due to slow tectonic forcings (Piromallo *et al.*, 1997), which will be described in the last section of this chapter.

Figure 6.8 reproduces the observed sea-level rates accounting only for the effects of active tectonics.

The pattern of sea-level change along the Adriatic coasts shows a high variability due to the geometrical complexity of the tectonic structures activated by the interaction between the Apennine mountain belt, the Adriatic plate to the east and the Alps to the north, as visualized in Figure 6.7. Active tectonics is responsible for significant rates of sea-level rise in the western coasts of the Adriatic sea. This sea-level rise is due to the downflexure of the Adriatic plate underneath the overthrusting Apennine belt, whose eastern border is represented by the thick line in Figure 6.7 separating the Tyrrhenian sector of the Italian peninsula to the west and the Adriatic domain to the east. Overthrusting of the Apennines is caused by the push of Africa from the south and by the decoupling, on geological time scales, of the western and eastern parts of the peninsula via a megafault. Figure 6.8 clearly indicates that active tectonics alone cannot explain archaeologically inferred sea-level records within the Adriatic. At Aquileia and the Istria promontory the tectonic model provides at most 0.1-0.2 mm/yr, which is at least 0.3 mm/yr lower than the observed

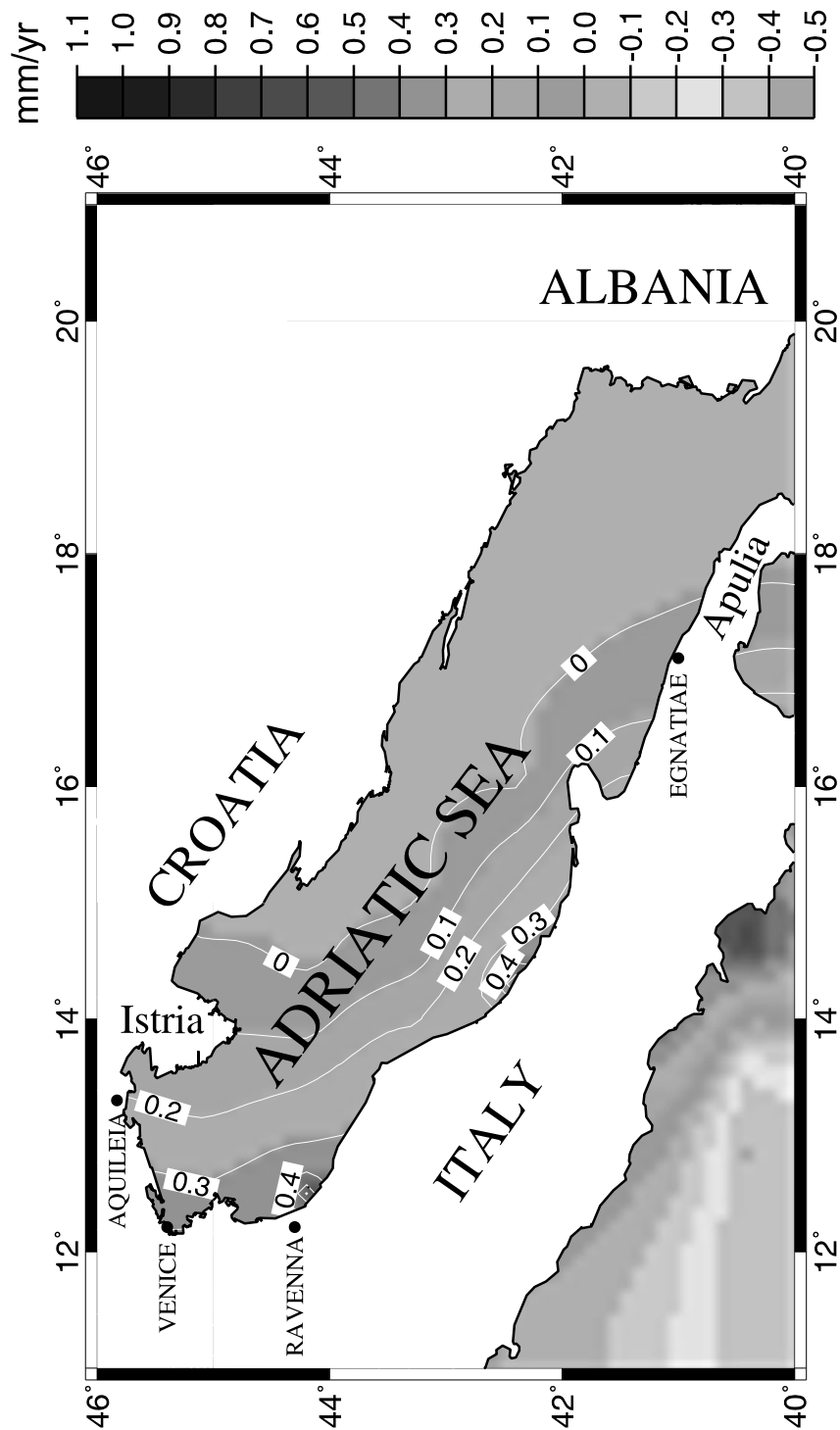


Figure 6.8. Present-day rates of sea-level change due to active tectonics in the Adriatic, the basin surrounded by Albania, Croatia and Italy. Redrawn from Figure 3 in Di Donato *et al.* (1999).

trend in this region (Schmiedt, 1979). The highest predicted sea-level rise in the Adriatic Sea due to active tectonics is 0.4-0.5 mm/yr, occurring in the Po delta plain, although at Ravenna even in this case the model predictions are significantly below the sea-level rise of 1.1 ± 0.2 mm/yr recorded at this site (Pirazzoli, 1998). The predicted sea-level rise at Ravenna achieves a maximum because the Adriatic platform experiences the largest subsidence at this site due to the arcuate geometry of the Apennines, and our predictions therefore reflect the three-dimensional geometry of the tectonic regime. In Apulia active tectonics yields a sea-level fall of about 0.1 mm/yr due to crustal uplift. This fall is consistent with the tectonic uplift of Apulia occurring on the time scale of millions of years (Pirazzoli, 1998).

Figure 6.9 deals with the effects of GIA. The most noticeable effect of the water load is a long-wavelength sea-level rise in the Adriatic Sea, increasing from 0.3 - 0.4 mm/yr in the northern part of the basin and along the coasts of Albania and Croatia to 0.5 - 0.6 mm/yr in Apulia. The large subsidence of the sea basin causes a slight uplift of the surrounding continental areas due to the flexural behavior of the lithosphere, which is thus ultimately responsible for the decrease in the sea-level signal in the eastern part of the Adriatic coast. Although the amplitude of this signal is somewhat dependent on the viscosity model assumed for the upper and lower mantle and on the thickness of the lithosphere, the pattern agrees well with previous results by Mitrovica and Davis (1995), with whom our results have been checked for stratified viscosity models in the Mediterranean up to harmonic degree 256. The GIA model alone cannot entirely explain the datum at Aquileia and the Istria promontory, since the model prediction reaches only 0.3 - 0.4 mm/yr at this site; the discrepancy between GIA prediction and observation is large at Ravenna, where the GIA simulation predicts rates of only 0.4 - 0.5 mm/yr. In Apulia, our GIA model correctly predicts the sea-level rise at Egnatiæ, indexEgnatiæ indicating that GIA is the dominant mechanism for sea-level change over the last thousand years at this site (as opposed to the above noted dominant tectonic effects in Apulia over time scales of millions of years). Figures 6.8 and 6.9 show that active tectonics and GIA are unable to account for the observed sea-level rise when taken separately.

Figure 6.10 shows the combined sea-level effects of active tectonics and GIA. In the northernmost part of the Adriatic Sea, where the Istria promontory and Aquileia are located, our combined models of GIA and active tectonics predict a sea-level rise of 0.4 - 0.5 mm/yr, in good agreement with archaeological data in Aquileia and, to a somewhat lesser extent, with the data in the Istria promontory. It is remarkable that such a conclusion on the necessity to incorporate both active tectonics and GIA in the northernmost sector of the Adriatic Sea can be drawn on the basis of the lower bound for sea-level rise of 0.4 mm/yr at the site of Aquileia. If we move to the south, along the western part of the Adriatic Sea

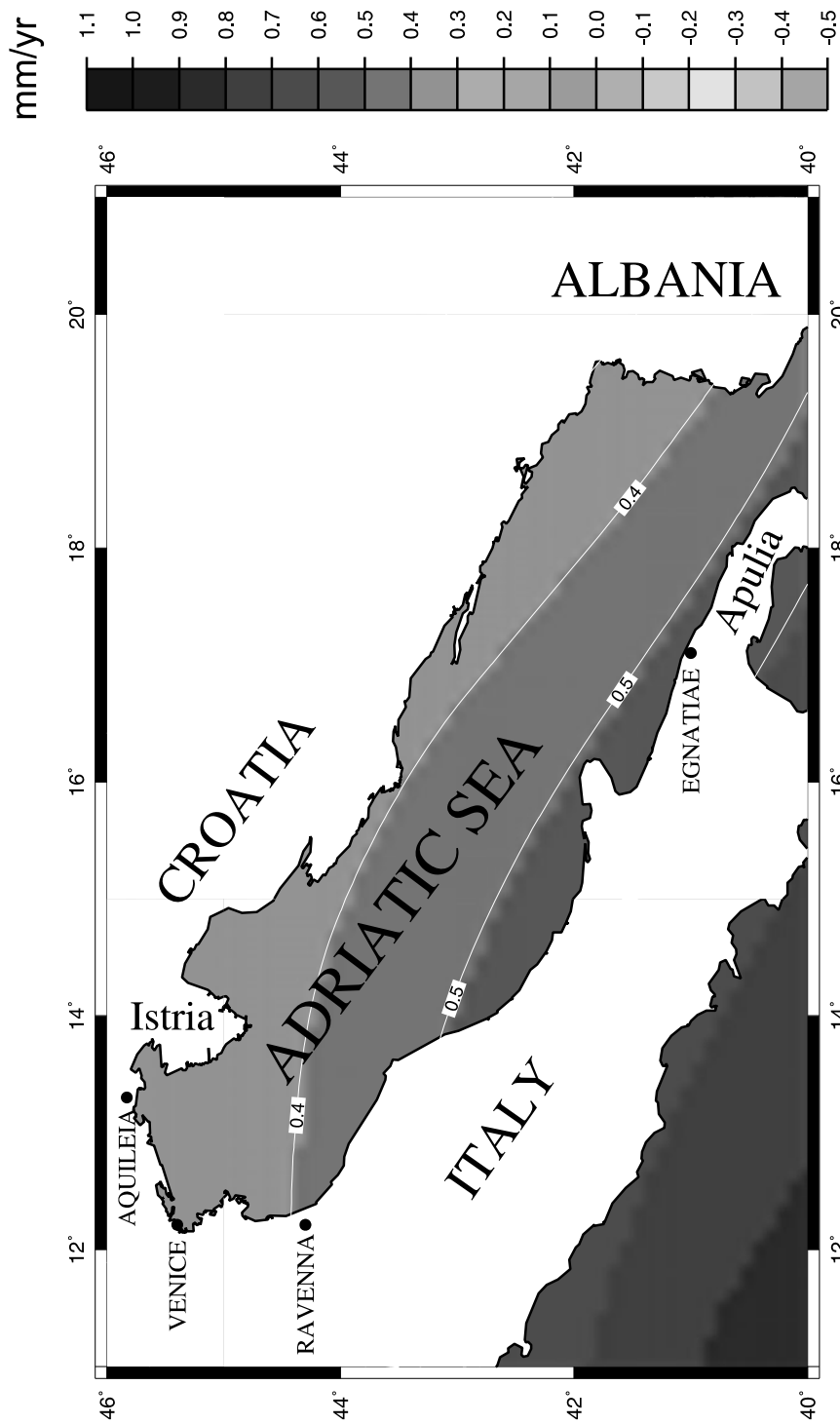


Figure 6.9. Present-day rates of sea-level change due to GIA in the Adriatic. Redrawn from Figure 4 in Di Donato *et al.* (1999).

in proximity to Venice and Ravenna, the modeling accounts for a substantial amount of the variation in the sea-level records from Venice to Ravenna; as noted already, sea-level records vary from 0.5 ± 0.2 mm/yr in Venice (Flemming, 1992) to 1.1 ± 0.2 mm/yr in Ravenna (Pirazzoli, 1998), in agreement with our results, varying from 0.75 mm/yr to 0.95 mm/yr between these two localities. Model results tend to underestimate the geographical variations in sea-level changes, perhaps due to small-scale tectonic features not taken into account in the modeling. It is remarkable, nevertheless, that the simulations reproduce the correct geographical trend, from which it is possible to conclude that the dependence of sea-level records on geographic location is due, in large part, to active tectonics. Compaction of the sediments, not modeled in the present analysis, could contribute to the sea-level record at Ravenna, and thus increase our prediction toward the higher bound on the observed trend. At Egnatiæ, a sea-level rise of 0.5 mm/yr agrees with the observed trend at this site.

Simultaneous modeling of active tectonics and Pleistocene ice-sheet disintegration in the central Mediterranean makes possible predictions on the trends in sea-level rise due to slow geodynamical processes. Consistency between data and model results in trends and magnitude of sea-level rise indicates that the major contributors to sea-level rise have been properly taken into account. A major finding is that contributions to sea-level rise due to GIA and active tectonics are comparable in the studied area, since active tectonics here is characterized by horizontal velocities at sub-centimeter level (Lanotte *et al.*, 1996). In other tectonic environments in the Mediterranean, with more significant horizontal velocities, active tectonics would provide higher sea-level signals, as in the island of Crete in the Aegean (Lambeck, 1995). The results in this section clearly show that it is possible to quantify the effects of active tectonics and GIA on sea-level rise in an area of great interest not only for the presence of cities like Venice and Ravenna, witnesses of a unique historical past, but also for the impact that climatic changes and anthropogenic activities, such as water and gas extraction, could have on the acceleration of sea-level rise.

4. SEA-LEVEL FLUCTUATIONS INDUCED BY POLAR WANDER

Glacial isostatic adjustment and tectonics are responsible for sea-level fluctuations at the regional scale, as shown in the previous section. Polar wander of a viscoelastic stratified Earth can induce, instead, global sea-level fluctuations comparable to the short-term component in the eustatic sea-level curves that show a characteristic pattern where sea-level rise and fall can occur simultaneously, depending on the geographical location. This simultaneous occurrence of sea-level rise and fall, visible as episodes of deposition or non-deposition of marine sediments in different parts of the world, has led to the proposal that at least some sea-level rises and falls must be global, or eustatic, in character.

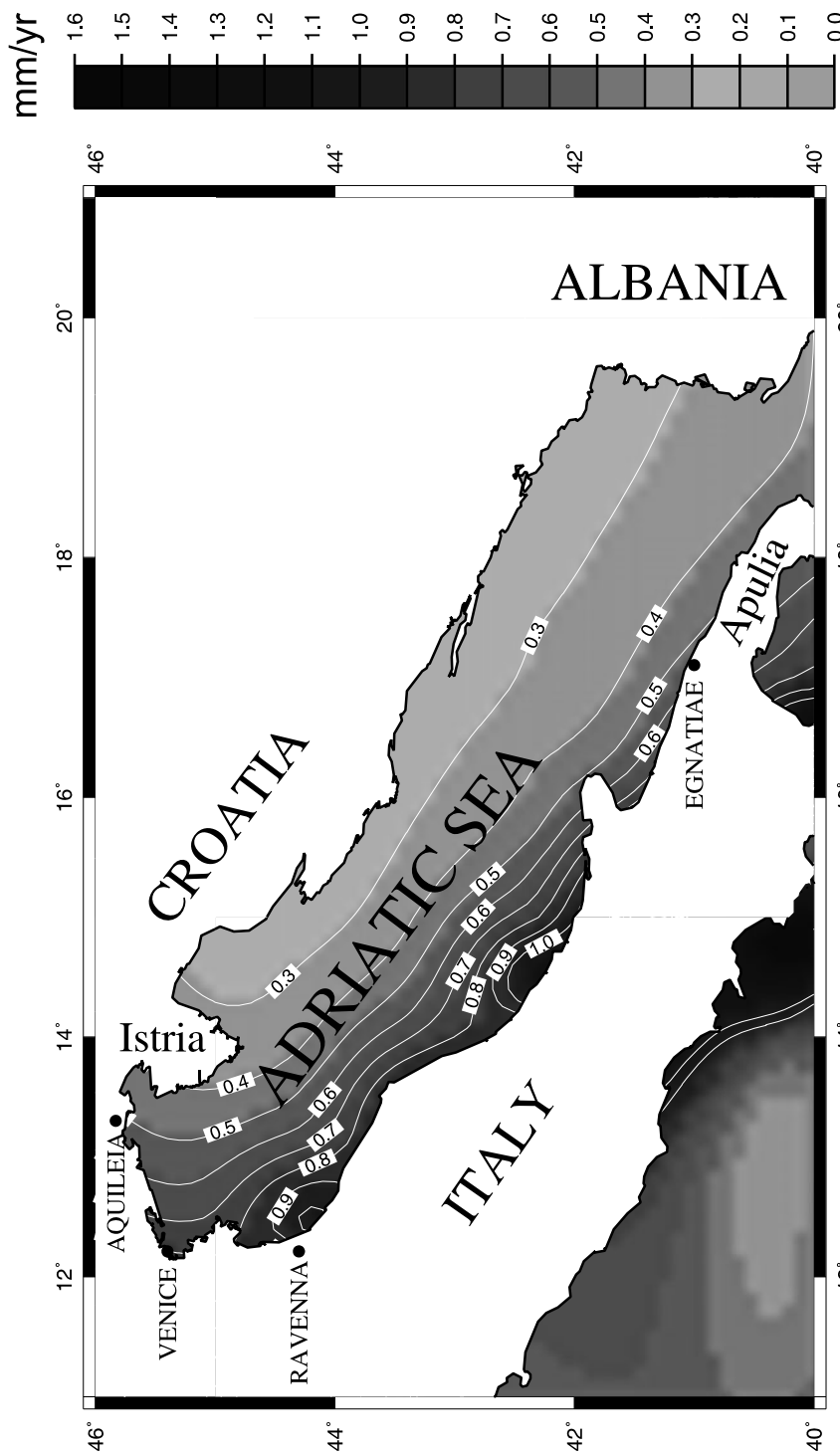


Figure 6.10. Present-day rates of sea-level change due to GIA and active tectonics. Redrawn from Figure 5 in Di Donato *et al.* (1999).

This simultaneous occurrence of sea-level rise and fall is in distinct contrast to the main assumption underlying the reconstruction of eustatic curves, i.e., that global sea-level events produce the same depositional sequence everywhere. This apparent contradiction may be due to the poor time resolution of the stratigraphic records in the distant past, which is comparable to the timescale of polar motion, and to non-uniform data coverage. The results presented in this section show that polar wander should be added to the list of geophysical mechanisms (the others are glacial instabilities, plate tectonic mechanisms, subduction, sea-floor spreading, and thermal and compaction-induced subsidence) that can control the third-order cycles in sea level. RSL fluctuations due to variations in the centrifugal force associated with long-term wander of the Earth's rotation axis can be predicted theoretically for radially stratified viscoelastic Earth models. The possible causes for these long-term displacements of the rotation axis can originate in the mantle, as suggested by the rotational responses of the Earth to the tectonic mechanisms described in Chapter 7, or can be caused by the surface redistribution of melt water from ice ages, as shown in Chapter 4. In this section the possible dynamic sources of polar wander are disregarded and attention is drawn instead to the effects on RSL induced by polar wander rates that agree with those induced by the geophysical processes considered in the other chapters of this book.

In the Laplace transform domain, the perturbation in the centrifugal potential depends, to first order, on the direction cosines of the rotation axis $m(s) = m_1(s) + im_2(s)$

$$\psi(s) = -(2\pi/15)^{1/2} \Omega^2 a^2 m(s) Y_2^1(\theta, \phi), \quad (6.21)$$

where $m(s)$ is the rotation rate of the Earth, a is the Earth's radius, $Y_2^1(\theta, \phi)$ is the spherical harmonic and θ, ϕ are the colatitude and longitude. Because our model is radially symmetric, and the distribution of the ocean is assumed uniform, this $l = 2, m = 1$ perturbative potential induces a signal in the relative sea-level fluctuation of the same angular degree and order. Neglecting the effects of self-attraction of the uniform ocean and of the geographical distribution of the coastlines, the RSL from equation (6.16) is then

$$RSL = (1 + k_2(s) - h_2(s))\psi(s)/g, \quad (6.22)$$

where k_2 and h_2 are the $l = 2$ components of the tidal Love numbers for the gravitational potential and vertical displacement, respectively, obtained from the boundary conditions at the Earth's surface given by equation (1.127) and g is the gravitational acceleration. Neglecting the self-attraction of the ocean has the effect of producing an increase of at most 10 per cent on our results. The term $1 + k_2(s)$ yields the deformation of the ocean surface relative to the Earth's center, whereas h_2 controls the vertical displacement of the sea-bottom.

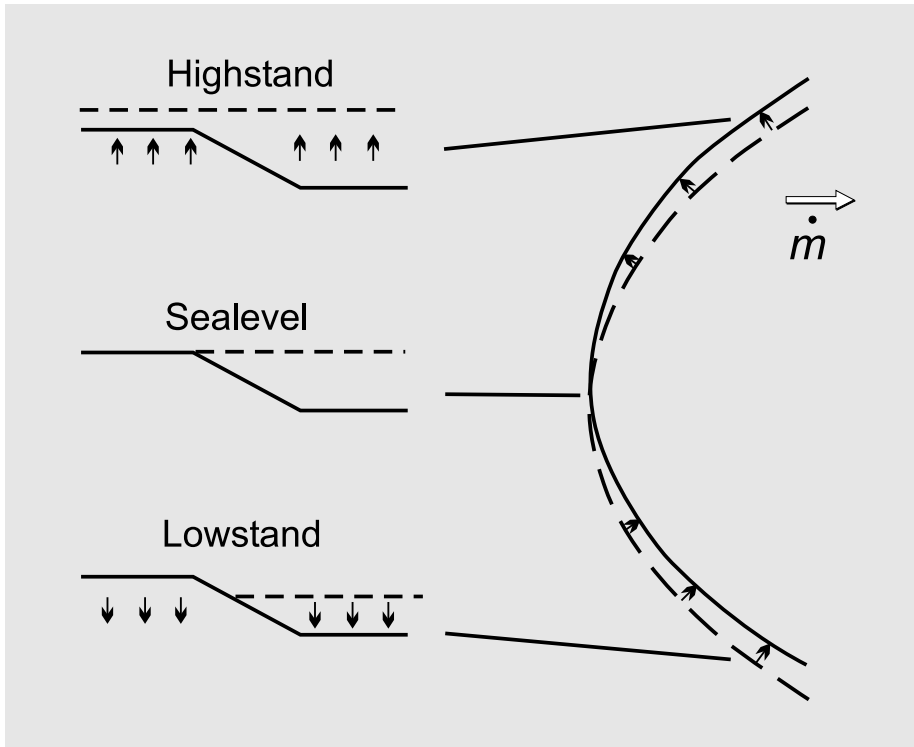


Figure 6.11. Pictorial representation of sea-level fluctuation induced by polar wander. The white arrow denotes the direction of polar drift at a rate $\dot{m} = 1 \text{ Deg/Myr}$. The solid and dashed curves represent the deformation of geoid and topography of the sea floor at time $t = t_0$, when the perturbation is initially imposed, and a subsequent time $t = t_1$. Sea level is unperturbed at the equator, but rises and falls occur in the Northern and Southern Hemispheres.

Owing to the lag between these two contributions, perturbations in the centrifugal potential induce sea-level fluctuations, as shown by the drawing in Figure 6.11. The white arrow indicates the direction of the polar wander, and the solid and dashed curves depict the shift of the equipotential surface (geoid). Highstands and lowstands are generated depending on the latitude and longitude of the observation points with respect to the polar motion. To quantify these sea-level fluctuations, it is necessary to make use of the spectral decomposition of the tidal Love numbers, which are recoverable by means of the normal mode theory, as shown by equation (2.2). The model used for the following simulations is a 4-layer model as given in Table 4.2, with viscosities ν_1 and ν_2 in the upper and lower mantle. The upper-lower mantle interface is located at a depth of 670 km and modeled in such a way as to simulate the mechanical behavior of phase-change or chemical transitions, which are assumed fully adiabatic and

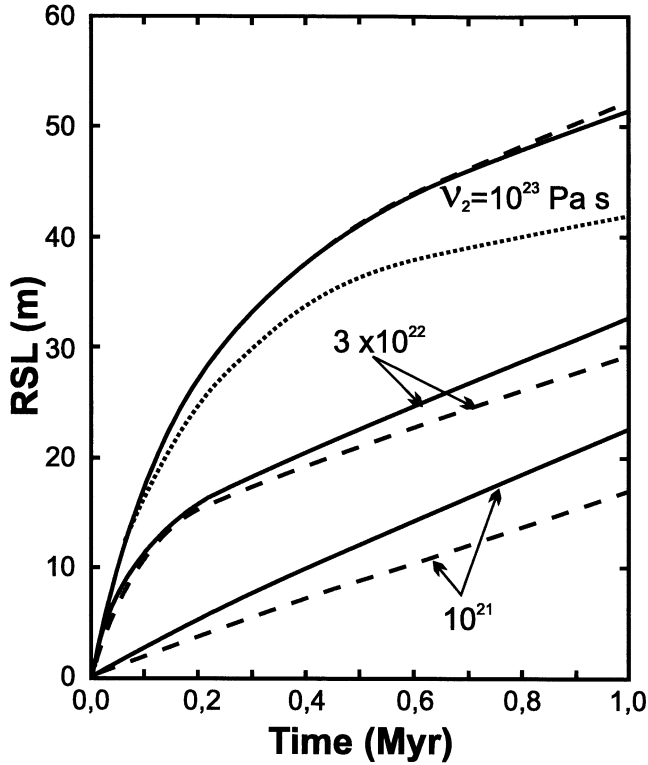


Figure 6.12. Time dependence of relative sea-level fluctuations at mid-latitudes ($\theta = 45^\circ$, $\phi = 0^\circ$), corresponding to $\dot{m} = 1$ Deg/Myr and polar motion toward Greenwich. The perturbative potential is applied at time $t = 0$. Solid curves correspond to chemically stratified models (fully non-adiabatic) and dashed ones correspond to fully adiabatic phase changes. Lithosphere thickness is 100 km, except for the dotted curve, which corresponds to 50 km. The upper mantle viscosity ν_1 is 10^{21} Pa s, and the lower mantle viscosity ν_2 is varied from 10^{21} Pa s (bottom) to 10^{23} Pa s (top).

non-adiabatic, respectively, as discussed in Section 4.7. The fully adiabatic mantle is modeled by deleting the M1 buoyancy mode, as discussed in section 4.7. Polar wander rate in the following simulations is 1 Deg/Myr, of the same order of TPW induced by post-glacial rebound or tectonic mechanisms (Chapters 4 and 7). The results for different polar wander rates can be obtained by means of linear scaling.

In Figure 6.12 the perturbative potential is applied at time $t = 0$; the evolution of the sea level at mid-latitudes, where the effects are larger owing to the latitudinal dependence of the perturbation, is depicted. Sea-level fluctuations are extremely sensitive to viscosity stratification and lithospheric thickness. These curves are characterized by transient behavior, more pronounced for high lower mantle viscosities, followed by a linear trend, which is connected to a constant

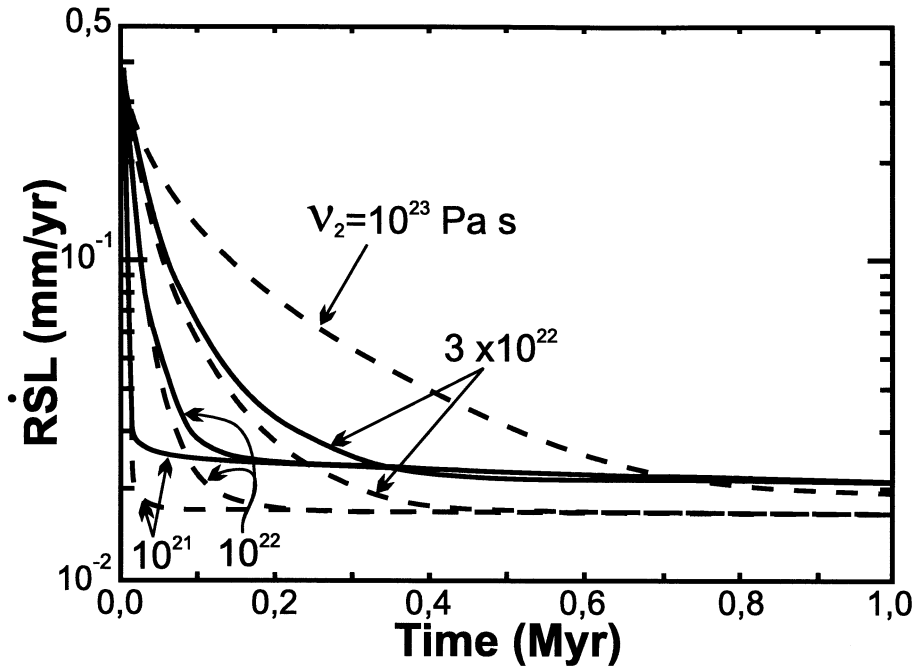


Figure 6.13. Time dependence of $R\dot{S}L$ in mm/yr. Solid curves correspond to chemically stratified models and dashed ones correspond to fully adiabatic phase changes. For $\nu_2 = 10^{23}$ Pa s, the chemical transition and phase-change models are indistinguishable. Results for $\nu_2 = 10^{22}$ Pa s are also shown.

rate of polar wander. For smooth viscosity contrasts, phase-change models (dashed curves) produce a smaller signal than chemically stratified ones (solid curves). For the chemically stratified models, deflections of the 670-km discontinuity induce a buoyant restoring force which inhibits viscous relaxation in the mantle. This, in turn, reduces the vertical uplift of the sea floor and helps to maintain the offset between sea-floor topography and the geoid. For high viscosity contrasts ($\nu_2 = 10^{23}$ Pa s), the stiffening of the lower mantle overcomes the dynamic effects associated with the nature of the 670-km discontinuity. Thus phase-change and chemically stratified models exhibit the same behavior. The dotted curve, corresponding to a model with a reduced lithospheric thickness of 50 km, shows the sensitivity of sea-level fluctuations induced by polar wander to variations in lithospheric thickness. High viscosity contrast models thus predict sea-level fluctuations in the order of several tens of meters on timescales of 1 Myr, which are comparable to the third-order cycle in the eustatic curves.

It is useful to analyze the behavior of the rate of sea-level fluctuations (Figure 6.13). Rates in the order of $0.05 - 0.1 \text{ mm yr}^{-1}$ are maintained for timescales of 0.5 Myr if the viscosity contrast is sufficiently high ($\nu_2/\nu_1 = 50$ to 100). For uniform models, or mild viscosity contrasts ($\nu_2/\nu_1 = 1$ to 10), rates of sea-level fluctuations decay in a few tens of thousands of years. After the decay of the initial transient, the rates reach a final value of 0.02 mm/yr, irrespective of the rheological stratification. As expected, phase-change models produce lower rates, except for high viscosity contrasts, in which case these models are indistinguishable from chemically stratified ones. We find that rates of sea-level fluctuations are around one order of magnitude smaller than those associated with vertical deformation of the sea floor, which are 1 - 1.5 mm/yr. The rates predicted by our model are of the same order of magnitude as the short-wavelength ones associated with temporal variations of the horizontal tectonic stress field in the lithosphere (section 6.3). From these results, it is clear that the proposed mechanism of polar wander is efficient in inducing sea-level fluctuation, especially during epochs of rotational instabilities, caused by mantle flows. Significant shift of the rotation pole occurred, for example, during the Late Cretaceous. The results shown in this section indicate that the highstand or lowstand system tracts observed in one part of the world are not necessarily correlated with the same depositional sequence in another geographic location. A highstand in a marine package in the Northern Hemisphere can, in fact, be coeval with a lowstand in the Southern Hemisphere at the same longitude if the only active mechanism that induced the eustatic sea level event was polar wander.

5. SEA-LEVEL CHANGES INDUCED BY SUBDUCTION

The effects of tectonics on relative sea-level changes have been shown at the regional scale. In this section we explore the effects of tectonics, in particular subduction, at the global scale. As in the case of the effects of rotation, we neglect the self-attraction of the oceans and the geographical distribution of the coastlines.

RSL is determined by the difference in height between the surface of the non-hydrostatic geoid and the surface of dynamic topography: in Figure 6.14 we illustrate the different contributions to the determination of RSL. The direct contribution of internal mass anomalies to the gravity potential is given by the term that would constitute the only contribution to the potential in the case of a rigid Earth (e.g., Lambeck, 1980). However, in a viscoelastic Earth, an additional compensating effect is induced by the deflection of the interfaces. The internal masses and the deflected interfaces together modify the sea water surface. In Figure 6.14, the resultant geoid and dynamic topography appear to be highly correlated both in amplitude and sign. However, this is only to be

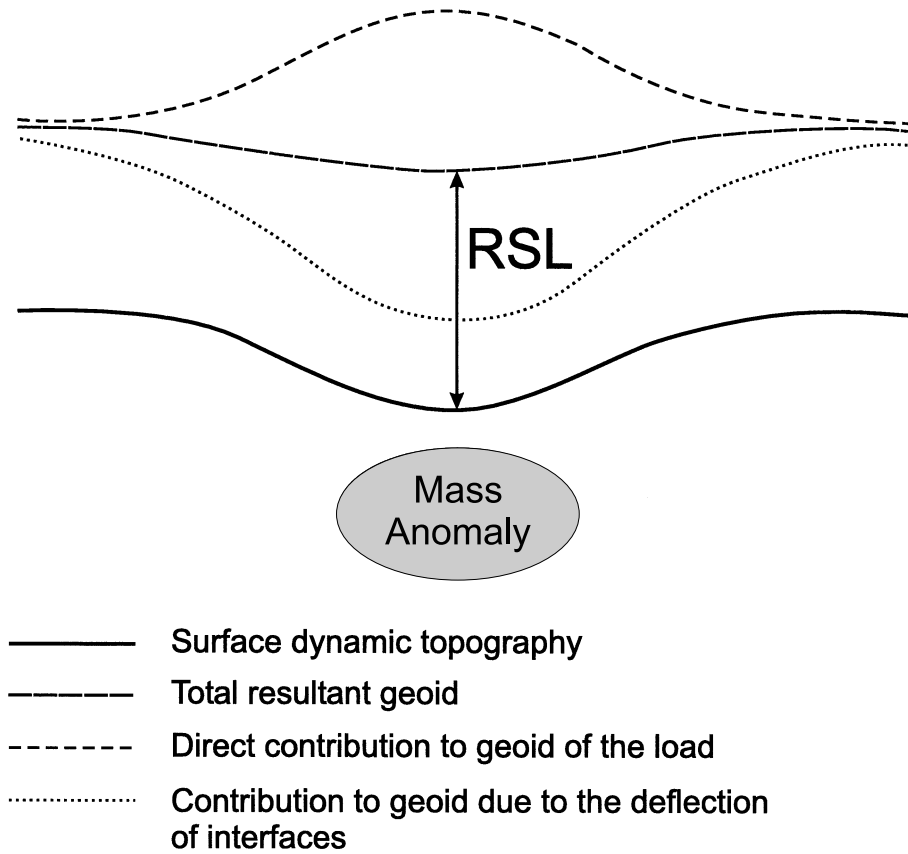


Figure 6.14. Schematic representation for the various contributions to RSL determination. RSL is determined by the offset in height between the surface of the resultant geoid and the dynamic topography, as shown by the arrow (adapted from Ricard and Vigny, 1989).

considered as a schematic representation and not as a general rule, since geoid and topography usually have much different amplitudes, as we will point out later on, and they may also be opposite in sign.

The generalized Love numbers used in this section are those appropriate for internal loads, equation (1.122) (Ricard *et al.*, 1992; Sabadini *et al.*, 1990; Sabadini *et al.*, 1993); they display an elastic response and various viscoelastic relaxation contributions, as in the case of the Love numbers for surface loads introduced in sections 1.8.1 and 1.8.2.

5.1 SEA-LEVEL VARIATIONS, GEOID ANOMALIES AND THE LONG-WAVELENGTH DYNAMIC TOPOGRAPHY

The study of geophysical observables related to the geoid and dynamic topography is fundamental for the understanding of relative fluctuations in the mean sea level. Several works have already analyzed in detail the geoid anomalies (e.g., Ricard *et al.*, 1992), while we will focus our attention on the dynamic topography in the present section. Since dynamic topography shows a strong degree 2 content well correlated with the non-hydrostatic geoid and with hot anomalies in the lower mantle (Cazenave *et al.*, 1989; Ricard and Vigny, 1989), the study of the degree 2 Love number for the topography turns out to be particularly interesting. Moreover, long-wavelength responses clearly illustrate the effects of mantle rheology and stratification on geoid and dynamic topography. We will develop our analysis up to a higher harmonic degree, $l = 20$, which is more appropriate to describe features such as subduction zones. The 4-layer earth model described in Table 4.2 is employed, with both phase-change and chemical interface at 670 km. A pure phase change interface implies a whole mantle convection, since the mantle material is allowed to flow across the boundary. A fully non-adiabatic density jump across the discontinuity (chemical interface), on the contrary, separates the mantle flow and generates a layered convection with two superimposed convective cells, as already discussed in Chapter 4 (e.g., Ricard and Vigny, 1989; Piromallo, 1993).

Figure 6.15 shows the degree 2 dynamic topography as a function of depth. We locate a point mass anomaly characterized by a Heaviside time-history at a depth d in the mantle in order to compare the effects of loads positioned at various depths. The different curves represent the dimensionless values of the topography computed at the time indicated (expressed in kiloyears) and normalized to the spectral amplitude of the load (Piromallo, 1993). In this figure the viscosities of the lower and upper mantles are both fixed at 10^{21} Pa s. No lithosphere is included in the results portrayed in the left column panels (Figures 6.15a, c, e, and g), while a 100 km thick elastic lithosphere is present in the right column (Figures 6.15b, d, f and h). The 670 km depth interface is characterized by a physical discontinuity in the four top panels and by a chemical change in the four bottom panels. While models a, b, e and f are isoviscous, models c, d, g and h are characterized by the presence of a sharp viscosity contrast between the lower and the upper mantle ($\nu_2/\nu_1 = 30$). The upper mantle viscosity is kept fixed at 10^{21} Pa s. The elastic response ($t = 0$) induces a negative surface topography and does not show large differences among the various models since the mass is not compensated, thus causing a downward deflection of the surface. As time increases up to the fluid limit ($t = \infty$), the characteristic features of the different models are clearly displayed.

For a physical discontinuity at 670 km (Figures 6.15a, b, c, and d) the dynamic topography is always negative. Topography vanishes for masses located at the core-mantle boundary in the asymptotic regime: for long time-scales, the masses are isostatically compensated by the density jump between the core and the mantle.

The time required to reach the isostatic regime is a few thousand years in the case of an isoviscous mantle (Figures 6.15a and 6.15b), while it rises to values in the order of 10^4 yr when the lower mantle is 30 times more viscous (Figures 6.15c, 6.15d). For a chemical transition (Figures 6.15e, 6.15f) masses located at 670 km depth are locally compensated, while those in the lower mantle cause a topographic high at the surface. In these latter models the asymptotic value is reached after a longer time interval due to the influence of the slower relaxation mode associated with the weak density jump at 670 km of depth. The viscosity increase causes a depression of the surface topography associated with lower mantle loads (Figures 6.15g and 6.15h). It is worth noting that topography values are larger in the models with the lithosphere due to the stress concentration in the vicinity of an elastic element. Note, too, that a mass, when situated close to the surface, induces a topography which is approximately independent of the mantle stratification: this corresponds to the usual concept of isostasy.

In Figure 6.16 the same kind of analysis is performed for RSL curves. It clearly appears that RSL shows a trend which is opposite to that of the dynamic topography at the surface. We generally observe, above a mass excess, a highstand of the sea level, except in the case of chemical transition, in which a lowstand is possible for lower mantle heterogeneities. The fact that RSL curves are roughly opposite to those of the topography shows that the contribution of the latter overwhelms the signal due to the geoid. Gurnis (1990b), by means of a one-dimensional kinematic model, studied the influence of the admittance (ratio of geoid to dynamic topography) on continental flooding. He has shown that the maximum allowable admittance is close to 0.11 for long wavelengths (Gurnis, 1990a). Our results reinforce these findings.

5.2 A SINGLE SINKING SLAB

We shall now consider RSL variations associated with a mass anomaly which mimics the effects of a slab sinking in the mantle at a constant velocity. The main assumptions of our model are: (i) the head of the mass sinks vertically, (ii) the mass extends from the surface to its head and thus increases with time, (iii) the sinking velocity corresponds to a typical velocity of subduction, (iv) the total mass of the anomaly is 2×10^{19} kg, corresponding to a slab pull of 5×10^{13} N/m along a trench 4000 km long (see Turcotte and Schubert, 1982). We consider a total mass of the anomaly, which is of the same order of magnitude as the ice sheets in the Northern Hemisphere during the Pleistocene

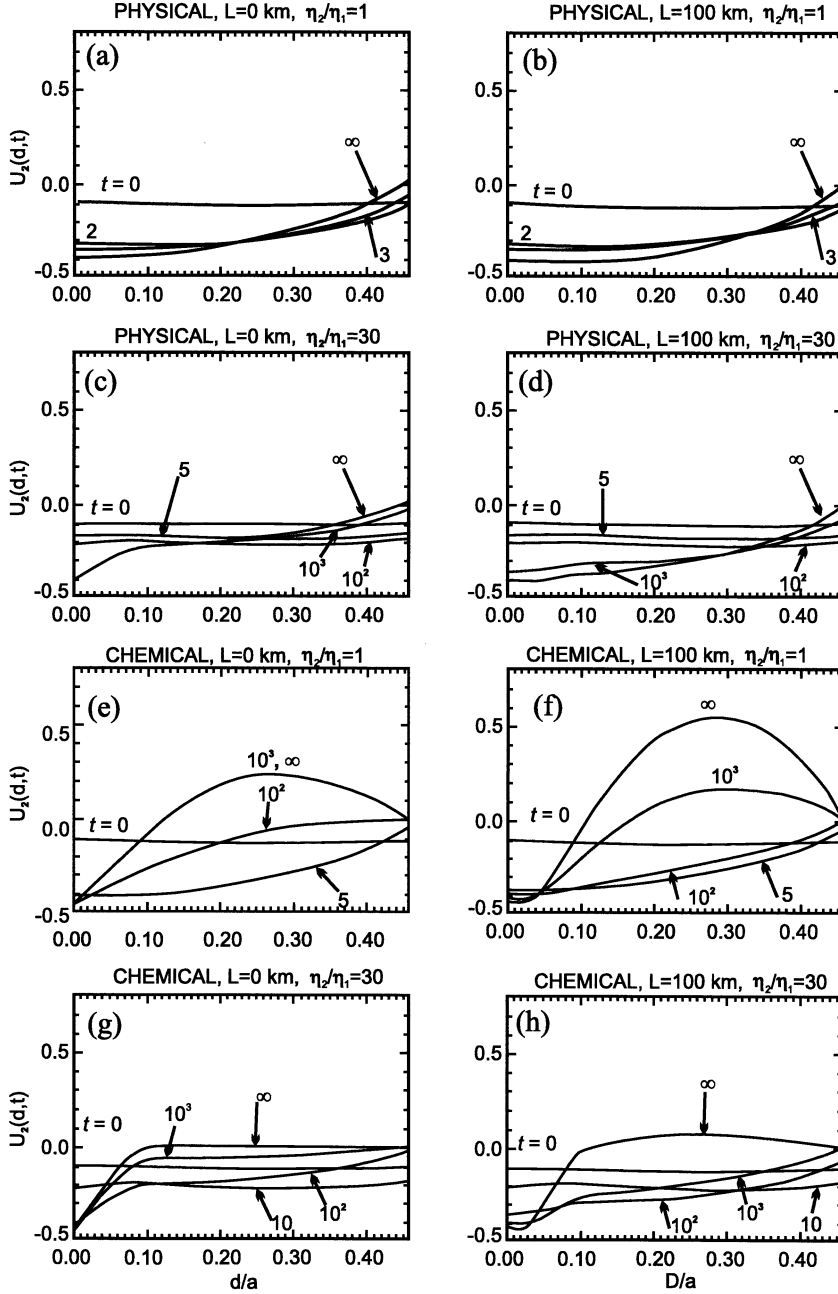


Figure 6.15. Time evolution for topography as a function of the point source depth d (normalized to the Earth radius a) and time t . A physical interface is assumed at 670 km of depth in the four top panels, while a chemical change is considered in the four bottom panels. A 100 km-thick lithosphere (L) is present in the right column panels. A viscosity contrast between the lower and the upper mantle ($\nu_2/\nu_1 = 30$) is assumed in models e, d, g and h. The curves have been computed at different times, expressed in kiloyears after the initiation of loading (Figure 2 in Piromallo *et al.*, 1997).

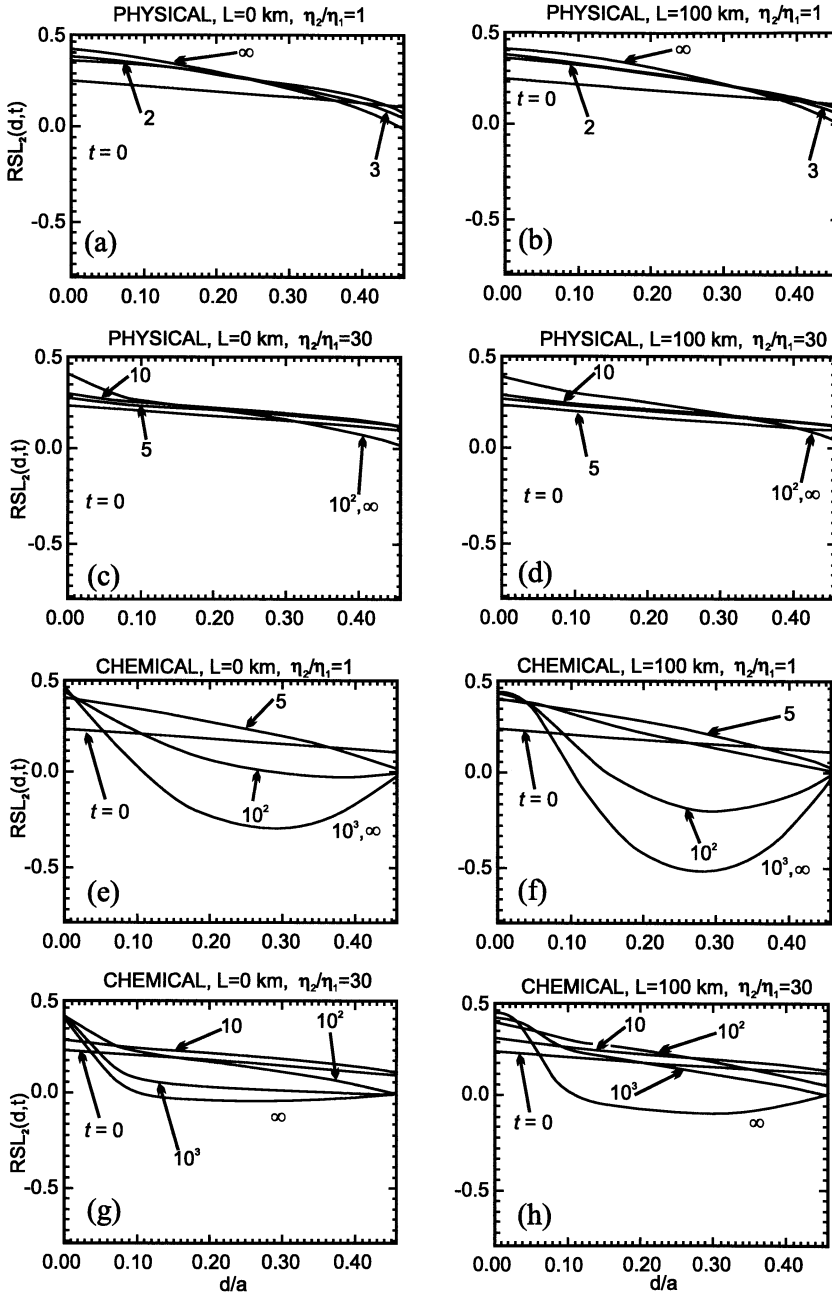


Figure 6.16. Time evolution for RSL (d, t) as a function of depth of the point source d (normalized to the Earth radius) and time t . A physical interface is assumed at 670 km depth in the four top panels, while a chemical change is considered in the four bottom panels. A 100 km-thick lithosphere is present in the right column panels. A viscosity contrast between the lower and the upper mantle ($\nu_2/\nu_1 = 30$) is assumed in models c, d, g and h. The curves have been computed at different times, expressed in kiloyear, after the initiation of loading (Figure 3 in Piromallo *et al.*, 1997).

deglaciation. This allows for a direct comparison of RSL variations due to various geophysical processes. For the sake of simplicity, we simulate the mass as a column composed of identical point elements progressively switching on at increasing depths from the surface down to the 670 km depth discontinuity.

In each panel of Figure 6.17 three values for the velocity of subduction are considered (2, 5 and 10 cm/yr). The $l = 2$ harmonic degree RSL is expressed in meters. We obtain only positive variations for RSL, thus showing that a subducting mass produces a highstand in sea level for each of our eight models. Another feature common to all models is the monotonically increasing trend of the three curves from the initial instant up to the point where subduction stops (with a gradient clearly depending on the subduction velocity). We observe an exponential decay in the condition of isostatic equilibrium (Figures 6.17e, f, g and h), governed by the relaxation of the chemical upper-lower mantle interface (see Spada *et al.*, 1992a; Ricard *et al.*, 1992). For the models with a physical interface (Figures 6.17a, b, c, and d) the asymptotic regime is reached on shorter time-scales. The fastest process (Figure 6.17a, b and d with $v = 10$ cm/yr) produces the largest variation, with 8 m of RSL in a time span of about 7 million years. In each figure, the three curves share the same asymptotic limit, independent of the velocity of subduction. For different models the fluid limit varies considerably, ranging from larger amplitudes for isoviscous models with a physical discontinuity (Figures 6.17a and b) to smaller ones for models with a chemical interface and no lithosphere (Figures 6.17e and g). In two-layered convective models, we notice that RSL varies rapidly at the beginning of subduction and progressively slows down as time increases. This trend is likely explained by the decrease in amplitude with depth of Green's function for dynamic topography, which mostly contributes to RSL as we already observed. RSL fluctuations are inhibited by viscosity contrasts, unlike in uniform models.

5.3 A DISTRIBUTION OF SLABS

RSL is now portrayed for a realistic distribution of slabs along the present-day convergent margins. The sinking slabs are discretized by means of point loads (slablets) subducting with the same constant velocity (following the approach in Ricard *et al.*, 1993). In the following simulations, it is assumed that no anomalies are present in the mantle before the initiation of subduction, which starts simultaneously along all the margins. All the slabs are then introduced at the same time, sinking into the mantle at a constant velocity, $v = 5$ cm/yr, and stopping when they have reached the 670 km interface. Figures 6.18 and 6.19 depict the RSL variations expanded up to harmonic degree $l = 20$, 15 Myr after the beginning of subduction. At this time, all the slabs have reached the 670 km depth and a steady-state situation is achieved by the complete relaxation of the viscoelastic flow. The harmonic degree $l = 20$ corresponds to a wavelength of about 2000 km, which allows a satisfactory spatial resolution. The two maps

display the results for models including a lithosphere and a physical change at the 670 km depth discontinuity.

In Figure 6.18 the mantle is isoviscous and corresponds to the model of Figures 6.15b and 6.16b, while in Figure 6.19 a $\nu_2/\nu_1 = 30$ viscosity contrast is assumed, as in Figures 6.15d and 6.16d. Since RSL is solely induced by slabs, the resulting patterns are strongly influenced by subduction margins, where large positive RSL variations are found, associated with small signals in the surrounding ocean basins. The contrast of viscosity within the mantle notably inhibits the variations in RSL. An upper bound for RSL variations induced by subduction yields values in the order of 0.1 mm/yr and is in agreement with previous studies by Gurnis (1992) based on two-dimensional models and comparable with the value attributed to changes in the large-scale tectonic regime of the Earth.

The contribution of the dynamic topography is by far more relevant than geoid variations in the assessment of RSL changes. Results from different models bear evidence of the impact, in terms of time-scales and amplitudes, of the overall viscosity profile and of the nature of the seismic discontinuities in the mantle in determining RSL. The presence of exponentially decaying viscoelastic relaxation times is clearly evident in the models with a chemical interface. RSL rates are enhanced by the elastic lithosphere, the physical interface at 670 km depth and uniform mantle viscosity. Subduction is an important geophysical mechanism controlling long-term sea-level curves in addition to glacial variations, polar wander, plate tectonic mechanisms, thermal and compaction-induced subsidence.

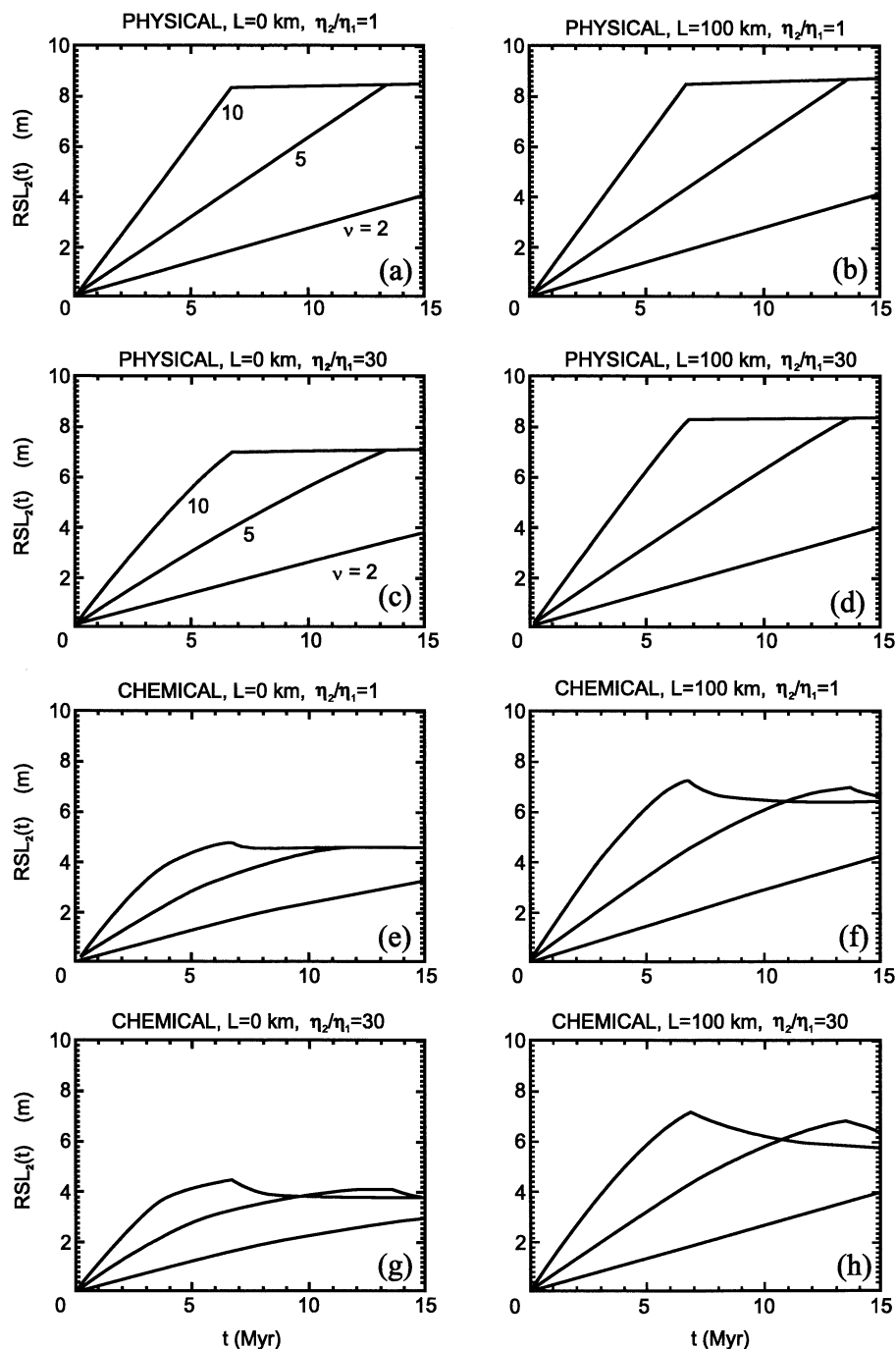


Figure 6.17. RSL variations, expressed in meters, as a function of time from the starting of subduction for a single slab sinking down to 670 km. The three curves for each panel correspond to the different values of average velocity of subduction indicated in the top left panel. The succession of models is the same as described in the previous figures (Figure 4 in Piromallo *et al.*, 1997).

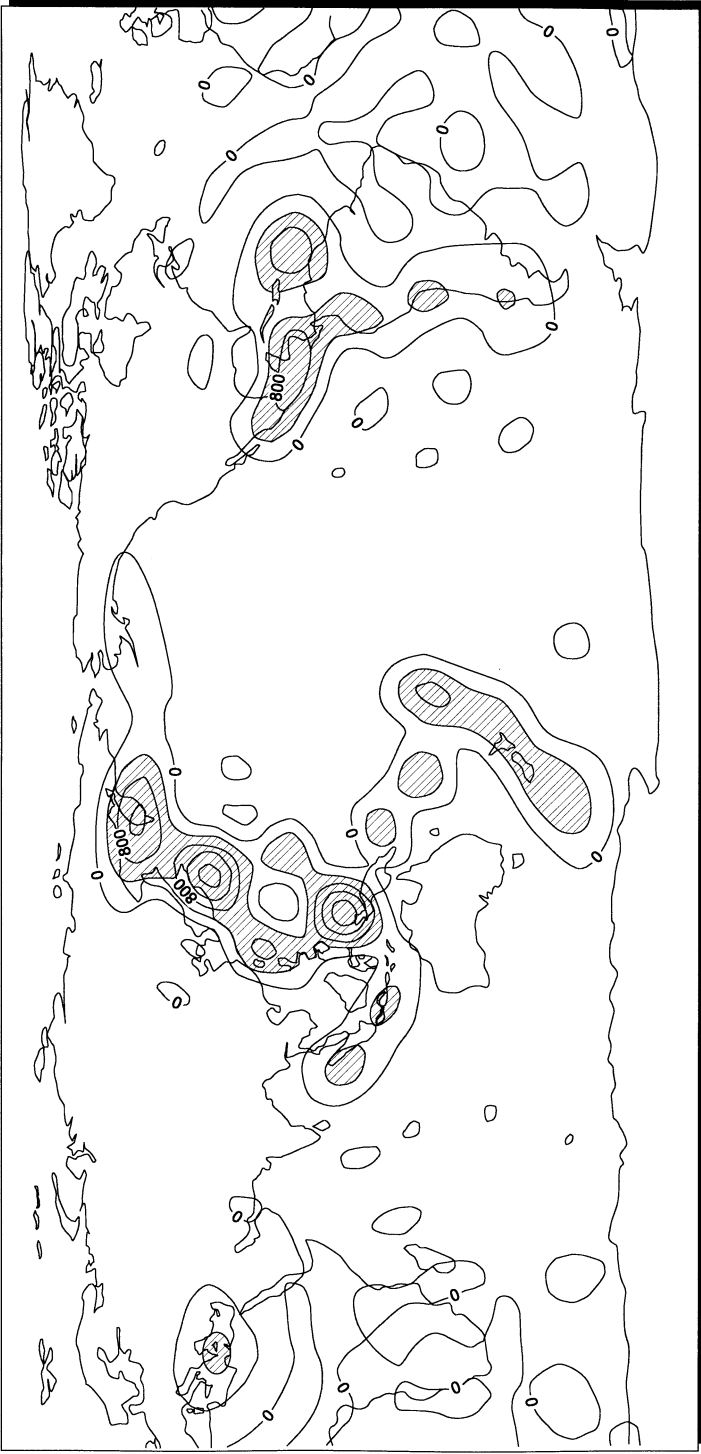


Figure 6.18. Map of the RSL variation induced by a realistic distribution of slabs up to degree 20, 15 Myr after the beginning of subduction. The model has a 100 km thick elastic lithosphere, a physical interface at 670 km and an isoviscous mantle (model b). Contours are 400 m apart. Redrawn from Figure 5 in Piromallo *et al.* (1997).

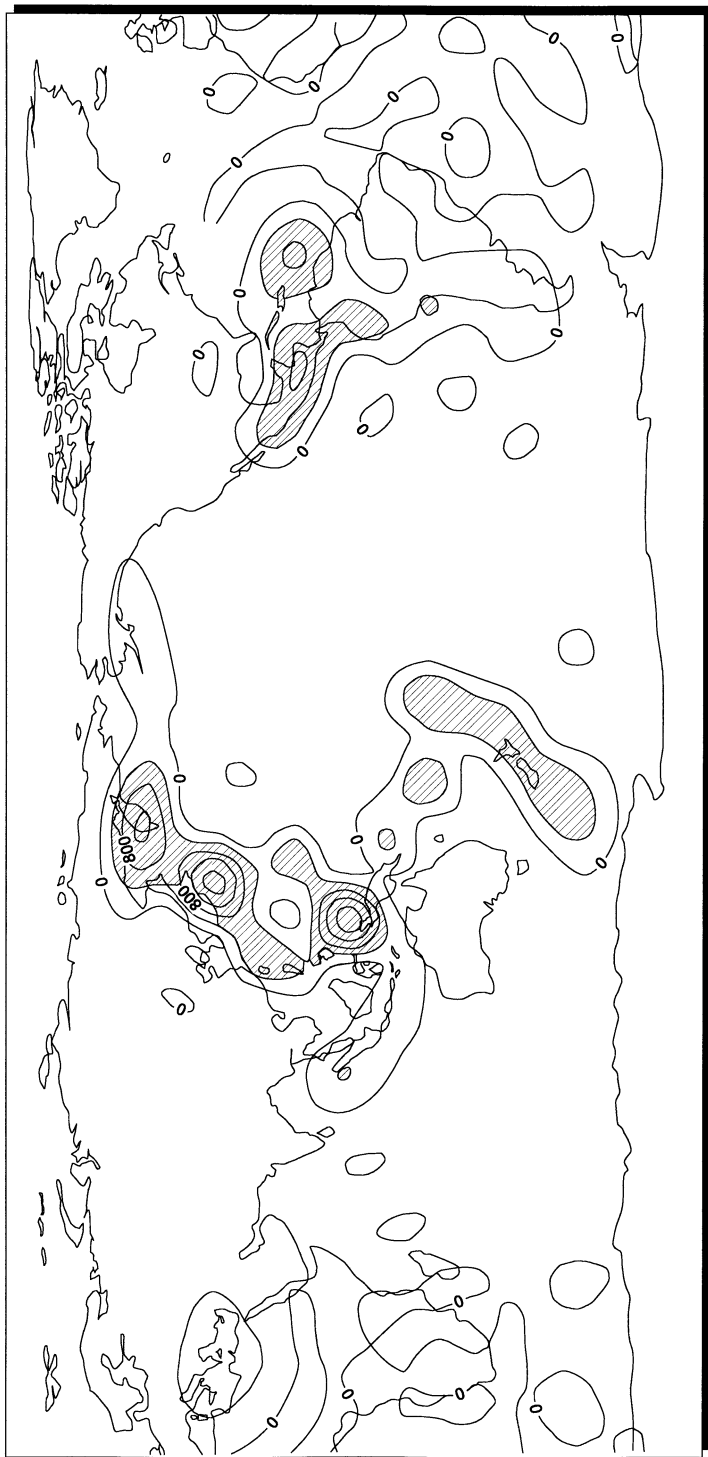


Figure 6.19. Same as in Figure 6.18, but with a viscosity contrast ($\nu_2/\nu_1 = 30$) between the lower and the upper mantle (model d). Redrawn from Figure 6 in Piromallo *et al.* (1997).

Chapter 7

TECTONICS AND THE ROTATION OF THE EARTH AND TERRESTRIAL PLANETS

1. MOUNTAIN BUILDING AND EARTH ROTATION

Contemporary secular changes in the rotation of the Earth have usually been attributed to post-glacial rebound and the present-day changes in the redistribution of ice and water over the surface of the planet (O'Connell, 1971; Nakiboglu and Lambeck, 1980; Sabadini and Peltier, 1981; Peltier, 1983; Gasperini *et al.*, 1986). Other analyses have shown the impact on TPW excitation of sources other than post-glacial processes.

Yukutake (1972) and Lefftz *et al.* (1991) have shown that processes acting at or near the core-mantle boundary can induce a significant secular drift of the polar axis and a non-negligible secular change in the rate of rotation.

Ricard *et al.* (1992), Spada *et al.* (1992a) and Ricard *et al.* (1993) have shown that subduction is likely to be the most important contributor to TPW excitation on million-year time scales because of the ability of deep-seated density anomalies to drive large amounts of flow in the mantle, which in turn is responsible for perturbations in the moment of inertia; these arguments are considered in the following sections.

Vermeersen (1993) and Vermeersen and Vlaar (1993) have proposed that non-glacially induced tectonic movements are also capable of inducing a significant contribution to the observed changes. They showed by a simple two-layer model that vertical lithospheric movements acting under non-isostatic conditions can be very effective in inducing both polar wander and non-tidal accelerations. However, their study lacked an assessment of the induced shift of the equatorial bulge. The mass displacements induce a torque on the equatorial flattening, which shifts the bulge by viscoelastic mantle flow. This induced process has only an effect on polar wander, not on changes in J_2 . In this model the effective density changes accompanying vertical movements are

restricted to the upper part of the lithosphere, where air or water is replaced by crustal material or vice versa, and the lower part of the lithosphere, where lithospheric material is replaced by mantle material or vice versa. As examples of operative tectonic mechanisms responsible for observed large vertical movements acting under deviations from isostasy, Vermeersen and Vlaar (1993) proposed mountain building and erosion in fold and thrust belts, lithospheric snapbacking resulting from detachment of subducting slabs and subsidence of oceanic basins. The main reason why these mechanisms can be effective is that compressive lithospheric stresses in combination with lithospheric and crustal rheologies prohibit the restoration of isostasy on time scales of post-glacial rebound. Rather than relaxing passively, as in models on post-glacial rebound, the mantle is envisioned as being subducted to the still acting lithospheric forces which maintain the deviations from isostasy on time scales exceeding those of post-glacial rebound. To illustrate the potential role of lithospheric processes in Earth rotation, Vermeersen and Vlaar (1993) estimated what the effects of the uplift of the Himalayas and Tibetan Plateau would be if the whole region is being uplifted by 5 millimeters per year. Within the framework of their model, which did not account for the effects of equatorial bulge readjustment, they found values of 20 per cent of the magnitude of the present-day true polar wander (the region is almost ideally situated for inducing changes in position of the rotation axis) and 4 per cent of the magnitude of the non-tidal changes in the second degree zonal geoid term J_2 (but one must be aware of the fact that the region is not ideally situated for inducing changes in this value; were the region located around the equator the influence would increase to about 15 per cent). In their models, Vermeersen and Vlaar (1993) assumed no compensation for uplift, subsidence or erosion. Though the mechanisms proposed can be very effective in establishing such a situation, there will always be an unknown amount of relaxation that diminishes the purported effects on the rotation and geoid signatures. Our approach does not take the lateral variations in the thickness of the lithosphere into account, which may play a role in the considered processes. A small change in geotherm or stress regime can alter the local (linear or non-linear) rheologies of crust and lithosphere drastically. Also changes in depth of the Moho, which could even increase by a factor of 2 in the case of lithospheric doubling, may impact the results. Apart from this, we model the whole region as one union, not discriminating between the mountain chain and the plateau.

The upper limits to the effects of mountain building are given by assuming that essentially no relaxation to the imposed load takes place; they are the values Vermeersen and Vlaar (1993) obtained for both the induced polar wander and the changes in J_2 . In order to investigate the effects of mantle relaxation, it is necessary to consider the temporal evolution of the root formation of the

modeled region, i.e. the Himalayas and Tibetan Plateau. Three root formation histories are considered: Heaviside, linear and exponential types of formation.

The readjustment of the equatorial bulge is taken into account in the calculations on polar wander. In the normal mode analysis carried out in this section 7.1 reworked from Vermeersen *et al* (1994), the Earth is divided into five layers by fixing the density contrasts: an effectively elastic lithosphere, a viscoelastic shallow upper mantle, transition zone, lower mantle, and an inviscid core, as in Table 2.1.

Sabadini and Peltier (1981) reported that the difference between employing the non-linear Liouville equations and the linearized version of them is less than 10 per cent for polar wander magnitudes of up to 15 degrees. As our calculated amounts of polar wander do not exceed this, we use the linearized version of the Liouville equations.

Though there are a number of uncertainties concerning the vertical movements and erosion rates and the relation between active lithospheric loadings and load relaxation, the time scales of up to a few million years on which the forcings we consider seem to be working open up new possibilities in studying the viscoelastic properties of the transition zone. The *M1* and *M2* modes associated with the 420- and 670-km boundaries are, in fact, the prevailing relaxation modes on these time scales.

1.1 UPLIFT HISTORIES

What has caused and is still causing the uplift of the Himalayas and the Tibetan Plateau, how large the uplift and erosion rates are and how great the deviations from isostasy were and remain hotly debated questions in the Earth sciences. That mountain building and plateau formation result from the collision between India and Asia some 40 to 50 million years ago is generally accepted, but with respect to the nature of the mechanisms responsible for the uplift the opinions differ. Harrison *et al.* (1992) review a number of these models, ranging from thrust of the Indian lithosphere beneath the entire area to crustal shortening and associated thickening of the crust until it has acquired a roughly double-normal thickness. Apart from these mechanisms, the uplift of the Himalayas during the last few million years has recently been seen as due to isostatic rebound in response to enhanced erosion rates (Molnar and England, 1990, and Burbank, 1992a). The emergence of the Pleistocene ice ages with the accompanying worsening in climate conditions are held responsible for this. At the moment it is still debatable, however, whether such a change in climate works only in a unilateral direction, i.e. whether ice promotes denudation because of glacier movements or diminishes erosion by forming a protecting layer.

In our modeling we will assume that the tectonic mechanisms are still active and that they cause most of the uplift. As Vermeersen and Vlaar (1993) have

pointed out, erosion can also be an effective mechanism in inducing rotational changes, even in the case when the erosion keeps pace with the uplift rate so that the mean altitude of the region is not changing (though the effects are then a factor of about one-sixth smaller since the effective mass changes are only due to the lower layer of Vermeersen and Vlaar's (1993) two-layer model). Erosion can even enhance the induced changes by promoting the deviations from isostasy in the case of an overcompensated light crustal root, which seems to be the case for the Himalayas (Basavaiah *et al.*, 1991). As to the temporal evolution of the uplift of the Himalayas and Tibetan Plateau, a canonical value for the time of initiation of uplift is 20 to 25 million years ago, a figure based on paleomagnetic data indicating that continental underthrust of the Indian plate under the Eurasian plate along the Main Central Thrust commenced during the early Miocene (Klootwijk *et al.*, 1985). The present-day mean altitude of the Himalayas and Tibetan Plateau is about 5 kilometers (Burbank, 1992b). Thus, if the uplift had been linear, the net uplift rates (uplift rates-erosion rates) from the time the uplift was initiated until present would have been 0.2 to 0.4 mm/yr. There are strong geological indications, however, that the uplift did not proceed in such a smooth linear fashion from the Miocene onwards. Zeitler (1985) presents uplift rates for the Himalaya ranges of northern Pakistan. He derived these rates from fission track and isotope dating. The rates remain generally below 1 mm/yr, except for the Quaternary. According to Zeitler, the Nanga Parbat-Haramosh Massif experienced an accelerated uplift rate that can be modeled by an exponential function: the uplift rate is equal to $5.0 \cdot \exp(-t/3)$, with t being the time before present in Myr, resulting in a present-day uplift rate of 5 mm/yr. From the fact that the uplift rates in northern Pakistan at least doubled during the late Tertiary, Zeitler concludes that this increase is not merely an isostatic response to unroofing, but must be ascribed largely to a further unspecified tectonic mechanism. One must be aware, however, that the values Zeitler reports derive only from a limited area. Copeland and Harrison (1990) state that the uplift of the Himalayas and the Tibetan Plateau did not proceed uniformly as in linear uplift models, or that it has taken place mostly from Pliocene time onwards, as in the exponential uplift model of Zeitler. They deduce this from sedimentary samples brought up during the Ocean Drilling Program Leg 116 in the Bay of Bengal. The sedimentation pattern is reported to be rather non-uniform, reflecting non-uniform erosion of the Himalayas, which in turn is assumed to be related directly to non-uniform uplift. This is further substantiated by a study of Amano and Taira (1992), who propose a two-phase uplift of the Higher Himalayas from the early Miocene to the Quaternary. From sedimentary strata of the Bengal Fan brought up during the same leg, they deduce that the Himalayas have undergone a surge of uplift between 10.9 and 7.5 million years ago and from 0.9 million years ago onwards. They claim that the enhanced deposition of the sediments during these time

intervals is not the result of channel migration or drainage basin change within a part of the Bengal Fan, but reflects a true rise in uplift rate with concomitant increased rates of erosion.

Although a lot is uncertain concerning vertical motions and erosion rates during geological and recent times, we model the low-density root formation of the Himalayas and Tibetan Plateau as if this region started to form this root 5 million years ago. We have chosen to model three kinds of root formation histories on the condition that the total amount of low density material would be equal to an uplift of 5 kilometers after 5 million years if the area were in Airy isostatic equilibrium. Apart from a time history in which root formation is proceeding in a smooth linear fashion and an exponential uplift history such as that advocated by Zeitler, we study a Heaviside type of root formation to approximate the pulselike uplift behavior as advocated by Copeland and Harrison and by Amano and Taira. We believe that these three cases in combination can adequately represent any geological uplift or root formation history that one wants to model.

1.2 CHANGES IN J_2 AND LINEAR ROOT FORMATION HISTORY

The formation of the mountain root is modeled by a negative load, shown in Figure 7.1a, approximated by a delta function forcing positioned at the center of the region.

This load originates from the mass deficit created by the formation of the low-density root responsible for the observed uplifting. The area extends from 70° to 95° East longitude and from 27° to 35° North latitude. The whole region is assumed to take part in the root formation uniformly. As the region is rather small in extent, the errors that this approximation for determining the effects on the second degree zonal component introduces are small. We can check the validity of this loading history by comparing the associated geoid anomaly and the observed one. A fully compensated topography of 5 km would imply a geoid anomaly of 25 m (Turcotte and Schubert, page 226). The observed one, for wavelengths corresponding to spectral coefficients higher than 20, is about 7 m, as shown in Figure 7.1b (based on Rapp and Cruz, 1986), implying that the missing 18 m can be ascribed to the overcompensation of the chain (Basavaiah *et al.*, 1991).

If the uplifting rectangular region is approximated by a circular one with an equivalent radius $R_d = 8^\circ$, a negative geoid anomaly $N = -20$ m requires a surface mass deficit σ of $2.8 \cdot 10^5$ kg/m² if we make use of the expression $N = 2\pi G R_d \sigma / g$ (see equations (5.71) and (5.143) in Turcotte and Schubert, 1982). The geoid anomaly, due solely to a topography of height $h = 5$ km with the same radius of the root and density $\rho_t = 2800$ kg/m³, which corresponds to a surface density anomaly of $1.4 \cdot 10^7$ kg/m², is 960 m, according to the previous

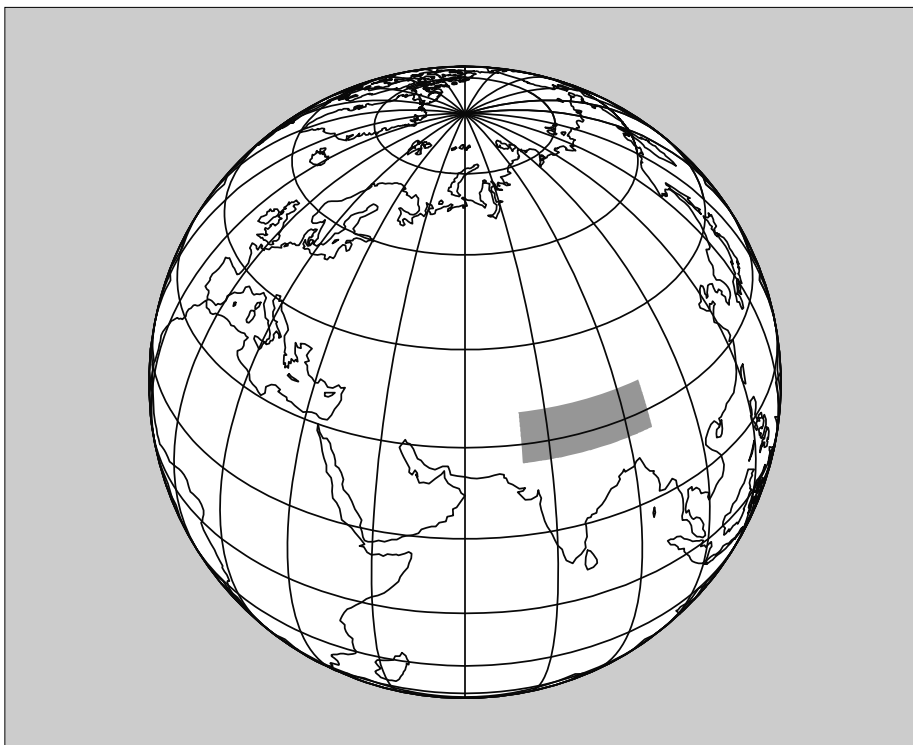


Figure 7.1a. Cartoon of the modeled Himalaya-Tibetan Plateau area, panel (a). The area extends from 70° to 95° East longitude and from 27° to 35° North latitude (Figure 1a in Vermeersen *et al.*, 1994).

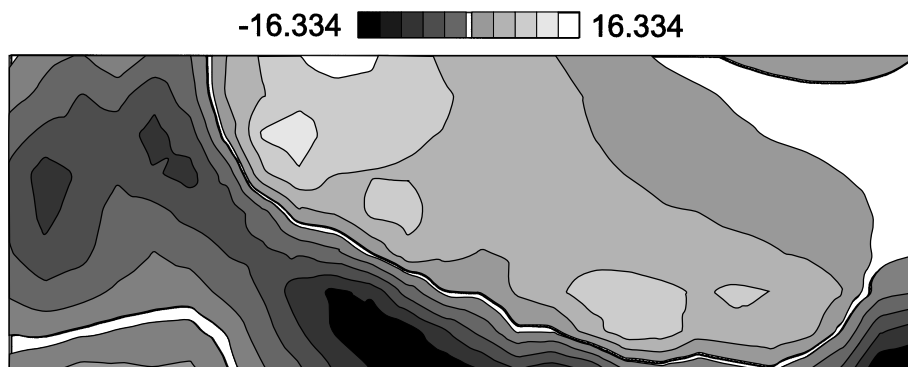


Figure 7.1b. This panel (b) portrays the geoid anomalies in meters for harmonic degrees larger than 20 for the modeled area (Figure 1b in Vermeersen *et al.*, 1994).

formula. From the ratio between the two surface density anomalies we thus find that the amount of overcompensation cannot be larger than 2 per cent. This amount of overcompensation can be imposed a priori in a model that does not account for mantle relaxation, or can be obtained in a self-consistent way by imposing the negative load of the mountain root on our viscously stratified earth model and then allowing for viscous relaxation in the mantle until this isostatic disequilibrium is maintained by the flexural properties of the elastic lithosphere of 100 km. Basavaiah *et al.* (1991) calculated, however, that the short wavelength free-air gravity anomalies over the Himalayas are better explained by a root which has a 30 per cent isostatic overcompensation, which is one order of magnitude larger than the value considered here. This estimated 30 per cent of overcompensation applies only to a more local area (the Himalayan mountain chain) than the $8^\circ \times 25^\circ$ area we are considering.

The degree 2 spectral coefficient of the perturbed geoid due to a delta function forcing on its surface is given in the Laplace domain by $\delta J(t) = -\frac{M_t}{M_E}(1 + k_2(s))P_2(\cos \gamma)$, where k_2^L denotes the loading Love number or perturbation in the geopotential in a-dimensional form, while γ denotes the angular distance from the place on the globe where the delta function load is. M_t and M_E denote the mass of the mountain root and total mass of the Earth, respectively, while P_2 is the second degree zonal harmonic. The second degree zonal harmonic δJ_2 is equal to the temporal convolution of $J(\gamma, t)$ with the low density root formation history $f(t)$:

$$\delta J_2(t) = \int dt' f(t') J(t - t') P_2(\cos \gamma), \quad (7.1)$$

in which the dimensionless root formation history $f(t)$ describes the formation of the root under the condition that $f = 0$ is equivalent to no root, while $f = 1$ is equivalent to the complete root.

In this first example, we model the root formation by assuming a linear growth starting at $t = 0$, so that at present we remain with a certain amount of isostatic disequilibrium due to the elastic lithosphere. Full mantle relaxation is allowed. At present, $t = t_0$, the function f has the normalized value of $f(t_0) = 1$. So:

$$f(t') = \frac{t'}{t_0} \quad \text{for } 0 \leq t' \leq t_0. \quad (7.2)$$

The changes in the degree 2 zonal harmonic are given by

$$\dot{J}_2 = -\frac{M_t}{M_E} \left(\frac{1 + k_e^L}{t_0} - \sum_{j=1}^M \frac{k_j^L}{s_j t_0} (1 - e^{s_j t}) \right) P_2(\cos \theta_0), \quad (7.3)$$

with the total equivalent mass $M_t = 4 \cdot 10^{19}$ kg. The simulations for this model are shown in Figure 7.2.

The values for $\dot{J}_2$ remain negative for the whole 5 million years. The increasing mass deficit close to the equator causes the Earth to become less elliptical. The decrease in the ellipticity is due to the fact that the relaxation cannot keep up with the low-density root formation. The absolute values of the induced J_2 remain two orders of magnitude too low for all modeled cases to have a significant influence on the present-day rate of change of J_2 . Similar conclusions can be drawn for the Heaviside and exponential growth of the load.

To summarize: if the mantle is merely relaxing passively, then on the time scales considered here (that is, exceeding those of post-glacial rebound) the values of $\dot{J}_2$ are orders of magnitude too low to be significant. However, as Vermeersen and Vlaar (1993) have pointed out, the lithosphere will at least partly inhibit this mantle relaxation due to in-plane stresses, so that the values for J_2 in Figure 7.2 must be interpreted as lower limits. The upper limits are given by assuming that there is no mantle rebound at all.

Another way to look at these results is the following: in the modeling we did not specify where the material causing the uplift of the Himalayas and the Tibetan Plateau comes from. The places where the compensating material originates are more likely to be wide-spread over the globe (with the majority most likely being beneath the oceans). These places are more likely to be subsiding under conditions where the mantle can relax passively, not prohibited by the lithospheric forcings in combination with intraplate stresses that are envisioned to be active in the Indian-South Asian collision zone. For these compensating regions, our approach is more applicable than in the region experiencing uplift. So, the induced changes in J_2 by the Himalayan and Tibetan Plateau uplift will probably be closer to the upper values derived by Vermeersen and Vlaar (1993) than to the lower values derived here (though the uplift of 5 mm/yr that Vermeersen and Vlaar (1993) used is certainly an upper limit as this would result in a topography of 5 km after 1 Myr), while on the other hand the changes induced in J_2 by the mass-compensating regions will likely be very close to the values of the simulations depicted in Figure 7.2. Much the same kind of reasoning can be set up for the erosion models.

1.3 POLAR WANDER AND HEAVISIDE, LINEAR AND EXPONENTIAL ROOT FORMATION HISTORY

The Laplace-transformed forcing function $\Phi_L(s)$ for a situation in which there is only mantle relaxation due to the changed centrifugal potential (no mantle relaxation due to the load) reads in the case of root formation

$$\Phi_L(s) = \mathbf{G}(\sigma, \theta, \phi) \tilde{f}(s), \quad (7.4)$$

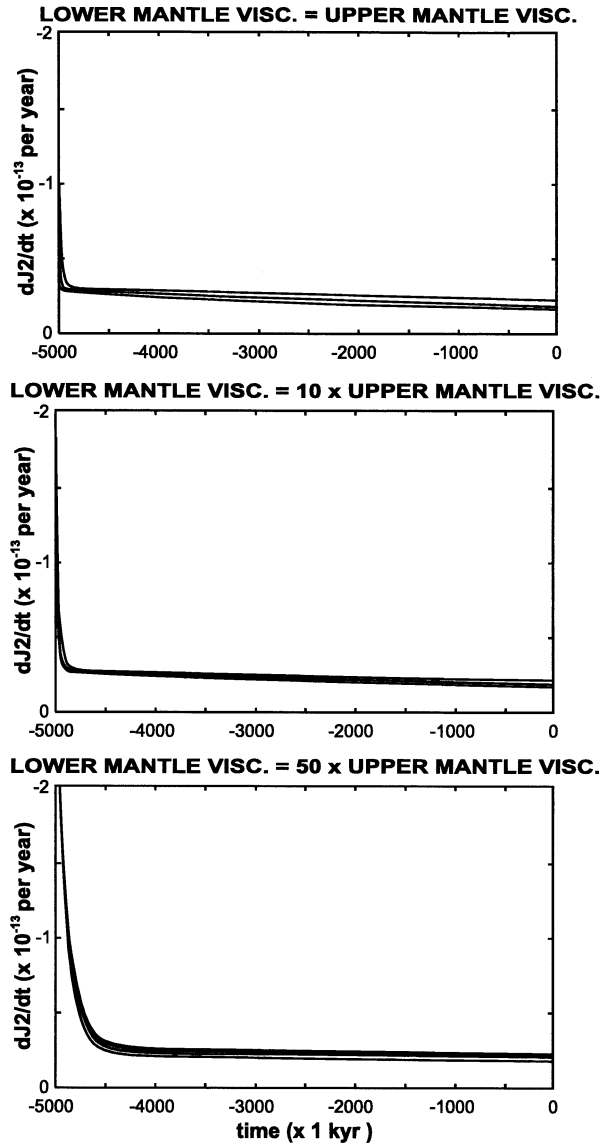


Figure 7.2. Change in J_2 by full mantle relaxation; linear case. The root started to form in a linear fashion five million years ago in such a way that at present an uplift of 5 km would be accomplished if the area was in Airy isostatic equilibrium. The shallow upper mantle viscosity is held fixed at 10^{21} Pa s, while the three panels show the cases for a viscosity of the lower mantle which are 10 times larger and 50 times larger than this value. Each panel contains four curves representing the cases for a viscosity of the transition zone that is five times, 30 times and 100 times larger than the viscosity of the upper mantle. In all cases the lowermost curve in each panel is for the equal viscosity case; the curves which show slower declining tendencies are for exceedingly higher viscosity contrasts between the transition zone and the shallow upper mantle. Redrawn from Figure 2 in Vermeersen *et al.* (1994).

with, for a load specified as a quadrilateral block (Vermeersen and Vlaar, 1993)

$$\mathbf{G}(\sigma, \theta, \phi) = -\frac{\sigma R^4}{3(C - A)}(\sin^3 \theta_2 - \sin^3 \theta_1) \times \\ (\sin \phi_2 - \sin \phi_1, -(\cos \phi_2 - \cos \phi_1)), \quad (7.5)$$

in which σ is the surface density, with $1.4 \cdot 10^7 \text{ kg/m}^2$ if we allow for relaxation in the mantle or $\sigma = 2.8 \cdot 10^5 \text{ kg/m}^2$ for the imposed isostatic disequilibrium.

We now analyze these two classes: in the first the mantle is relaxing only to the tidal-effective forcing but not to the load. In the second the mantle is relaxing to the load and tidal-effective forcing.

For the case in which the mantle relaxes to the induced change in centrifugal potential, but not to the load (equation (3.57) without k_e^L, k_j^L)

$$\tilde{\mathbf{m}}(s) = -i\sigma_r \mathbf{G}(\sigma, \theta, \phi) \left(\frac{A_0}{s} + \sum_{i=1}^M \frac{A_i}{s - a_i} \right) \tilde{f}(s), \quad (7.6)$$

while for fully operating mantle relaxation the complete equation (3.57) must be dealt with. The three cases for the temporal evolution of the root will be considered: root formation in a very short time interval which can be approximated by a Heaviside function, linear and exponential root formation.

For the Heaviside case, we consider first the case of the mantle relaxing only to the readjustment of the equatorial bulge and then full mantle relaxation, even to the surface density anomaly. If we take the root as being instantaneously formed at $t = 0$, then the function $f(t)$ has the form $f(t) = H(t)$, with $H(t)$ being the Heaviside function. From equation (7.6) we get in the time domain for $t \geq 0$

$$\mathbf{m}(t) = -i\sigma_r \mathbf{G}(\sigma, \theta, \phi) \left(A_0 t + \sum_{i=1}^M \frac{A_i}{a_i} (e^{a_i t} - 1) \right). \quad (7.7)$$

Figure 7.3 shows the results of the calculations with the same rheological parameters as were used in the modeling on $\dot{J}_2$. The three panels in Figure 7.3 show the temporal development of the magnitude of the polar wander $m = \sqrt{m_1^2 + m_2^2}$ as a function of the viscosity contrast between the lower and shallow upper mantle and 2 per cent of uncompensation.

In each of the panels the situation is depicted as a function of the viscosity contrast between the transition zone and the shallow upper mantle. It is clear from the third panel that for high viscosity contrasts between the lower and shallow upper mantle the polar wander is small, irrespective of the viscosity contrast between the transition zone and the shallow upper mantle. This is in agreement with what Spada *et al.* (1992a) found from analyzing the effects

of long-term changes in subduction on the rotational axis. For small viscosity contrasts between the lower and shallow upper mantle, however, the viscosity of the transition zone has a strong influence on the induced amount of polar wander. The first panel shows that if the lower mantle had the same viscosity as the shallow upper mantle, the induced polar wander would be about two times larger than the case of a mantle having a uniform viscosity and a transition zone having a viscosity 30 times higher than the rest of the mantle. This panel clearly shows the transition between the dominant long-term relaxation modes $M1$ and $M2$ and the short-term relaxation ones. The polar wander rates are greatest during the first few hundreds of thousands of years and become constant thereafter, being completely determined by the long-term modes. Long-term rates for the uniform mantle are at most 0.3 deg/Myr, which is lower than polar wander rates due to deep-seated density anomalies associated with subduction, which are able to drive a larger amount of flow in the mantle and perturbations in the moment of inertia (see panel (f), in Figure 7, of Ricard *et al.*, 1992 or Figure 7.11 in this chapter). One order of magnitude of viscosity increase in the lower mantle decreases polar wander velocities at 0.2 deg/Myr.

For the case of mantle also relaxing also to the load, the polar wander from equation (3.61) for $t \geq 0$ is given by

$$\mathbf{m}(t) = -i\sigma_r \mathbf{G}(\sigma, \theta, \phi) \left(A_0^* t + \frac{\gamma_i}{a_i} (e^{a_i t} - 1) \right). \quad (7.8)$$

In this case and in the following full mantle relaxation ones the geoid anomaly carried by the degree 2 component is only 1 m in the steady-state regime, which is consistent with the observations. Figure 7.4 shows the temporal development of this fully operating mantle relaxation case.

Apart from the absolute magnitudes, the form of the four cases in each of the three panels does not differ very much from the former cases in which no mantle relaxation to the load was assumed to take place. It is interesting to note the similarity between Figures 7.3 and 7.4, in both the transient and steady-state regimes. The similar behavior during the transient simply reflects the fact that this is controlled by the rotational relaxation times a_i and readjustment of the equatorial bulge, with essentially minor differences in the strengths A_i and γ_i . The similarity of the steady-state regimes means that the assumed 2 per cent of uncompensation in the first model is consistent with the flexural properties of the 100-km lithosphere.

The role played by the different relaxation modes in the readjustment of the equatorial bulge is considered now in Figure 7.5. It is clear that if the lower to shallow upper mantle viscosity ratios are not too large (about 10 or less), the amount of polar wander becomes very sensitive to the properties of the transition zone. The contributions of the long time-scale relaxation modes $M1$ and $M2$ on the induced amount of polar wander is sketched in Figure 7.5. The

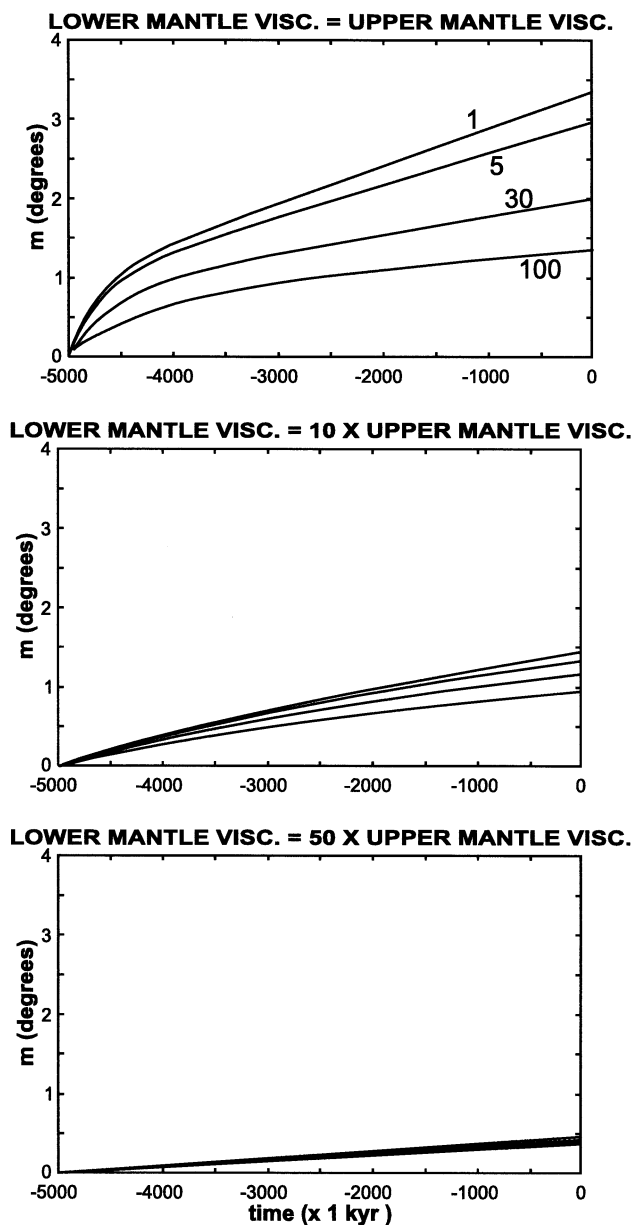


Figure 7.3. Polar wander by tidal-effective relaxation only; Heaviside case. The 2 per cent root was invoked instantaneously five million years ago and is assumed to keep the mantle from relaxing. The magnitude $\sqrt{m_1^2 + m_2^2}$ of the induced polar wander is plotted as function of time before present. The viscosity of the shallow upper mantle is again held fixed at 10^{21} Pa s for all cases. The four numbers for each curve in the first panel denote the values of the viscosity contrasts between the transition zone and the shallow upper mantle. The relative position of the four curves in the other panels is the same. Redrawn from Figure 3 in Vermeersen *et al.* (1994).

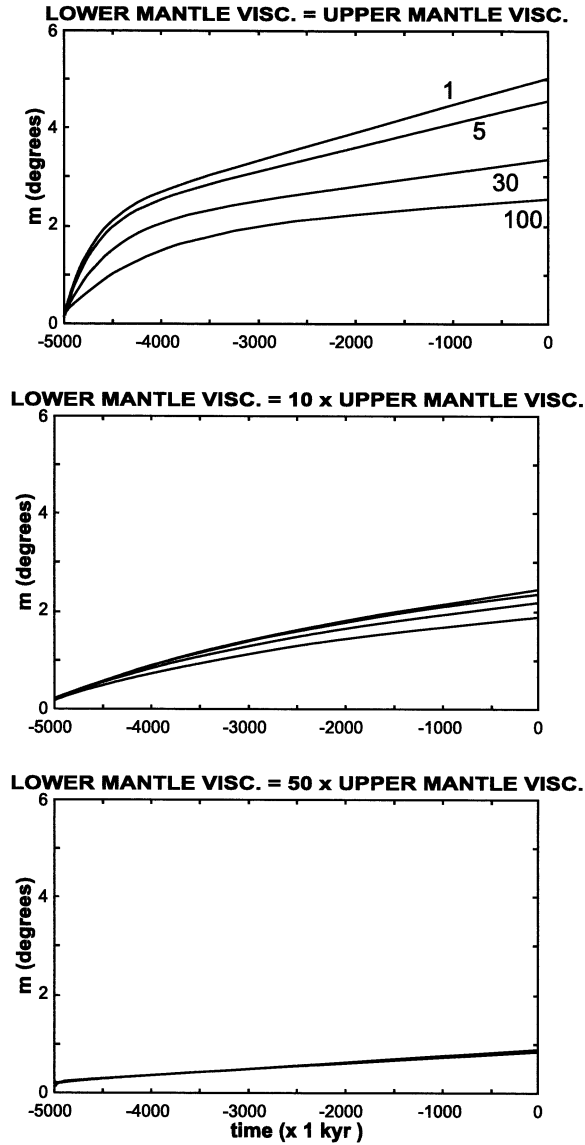


Figure 7.4. Polar wander by full mantle relaxation; Heaviside case. Both load and tidal-effective relaxation are operative. The formation of the root took place instantaneously five million years ago. The same remarks concerning the viscosity contrasts apply as in Figure 7.3. Redrawn from Figure 4 in Vermeersen *et al.* (1994).

four panels in this figure depict the evolution of total polar wander and the contributions from $M1$ and $M2$ as a function of four different mantle viscosity stratifications.

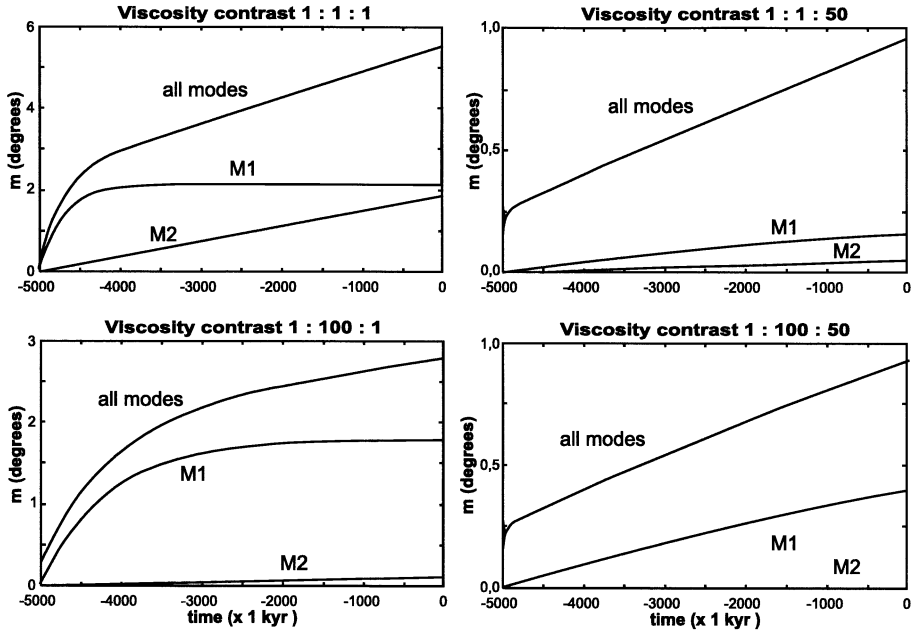


Figure 7.5. Relative importance of the $M1$ and $M2$ modes. The viscosities on top in the four panels indicate the viscosity of the shallow upper mantle, transition zone, lower mantle, in units of 10^{21} Pa s. The curves denoted 'all modes' depict the total amount of induced polar wander by the Heaviside case in which there is both load and tidal-effective relaxation. The curves denoted 'M1' and 'M2' give the contributions from the $M1$ - and $M2$ -mode relaxations, respectively. Redrawn from Figure 5 in Vermeersen *et al.* (1994).

In all four cases the $M1$ mode, which is related to the relaxation of the 670-km discontinuity, has more strength than the $M2$ mode. Only for times exceeding 5 million years has the $M2$ mode more strength (modeling results show that this will be the case for 5.8 Myr after the triggering of the Heaviside function for the uniform mantle case in the first panel). The reason for this is the long time scales over which the $M2$ relaxes. It is clear from all four panels that the $M2$ mode shows no sign of diminishing in strength: its temporal behavior is almost linear.

Figure 7.6 shows that the relaxation time of the $M2$ mode is one to two orders of magnitude larger than the $M1$ mode for all modeled values of the viscosity contrasts. This explains why the trend of the contribution of the $M2$ mode in Figure 7.5 seems linear: 5 million years are short compared to the $M2$ relaxation times, which exceed 100 million years.

That the amount of polar wander is not very sensitive to changes in density contrast between the upper mantle and the transition zone, or between the lower

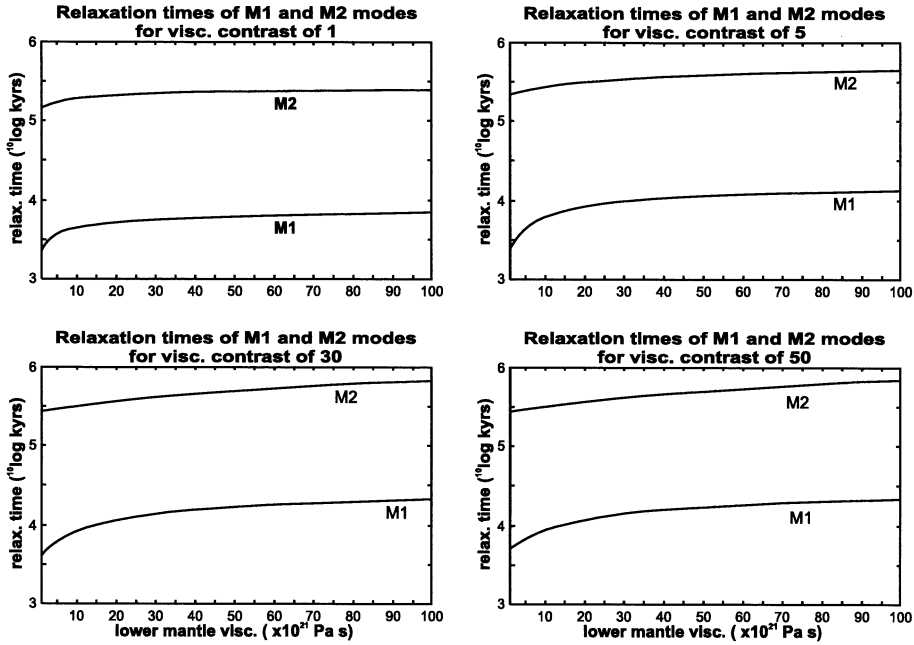


Figure 7.6. Relaxation time scales of the $M1$ and $M2$ modes. The relaxation times are logarithmically plotted as a function of lower mantle viscosity. The viscosity contrast referred to at the top of each panel is the viscosity contrast between the transition zone and the shallow upper mantle. Redrawn from Figure 6 in Vermeersen *et al.* (1994).

mantle and the transition zone, is shown in Figure 7.7. One of the Heaviside uplift cases in Figure 7.4 is chosen: the viscosity contrast between the transition zone and the shallow upper mantle is held fixed at 5, while that between the lower and the shallow upper mantle is 10. In the top panel in Figure 7.7 the polar wander is plotted for three values of the density of the lithosphere and upper mantle: 4000 kg/m^3 , 4120 kg/m^3 as in Table 2.1 for the fixed density contrasts and 4200 kg/m^3 . In the bottom panel the same is done for the values of the lower mantle: 4400 kg/m^3 , 4508 kg/m^3 (Table 2.1, fixed density contrasts) and 4600 kg/m^3 .

It is obvious from these curves and from comparison with Figure 7.4 that changing the density contrast affects the curves far less than changing the viscosity contrasts.

For a linear rate of root formation, which is supposed to start at $t = 0$, $f(t)$ can be written as $f(t) = t/t_0$ if the time-span to the present amounts to t_0 under the condition that at present f must be equal to one. The time history for this linear root formation rate reads

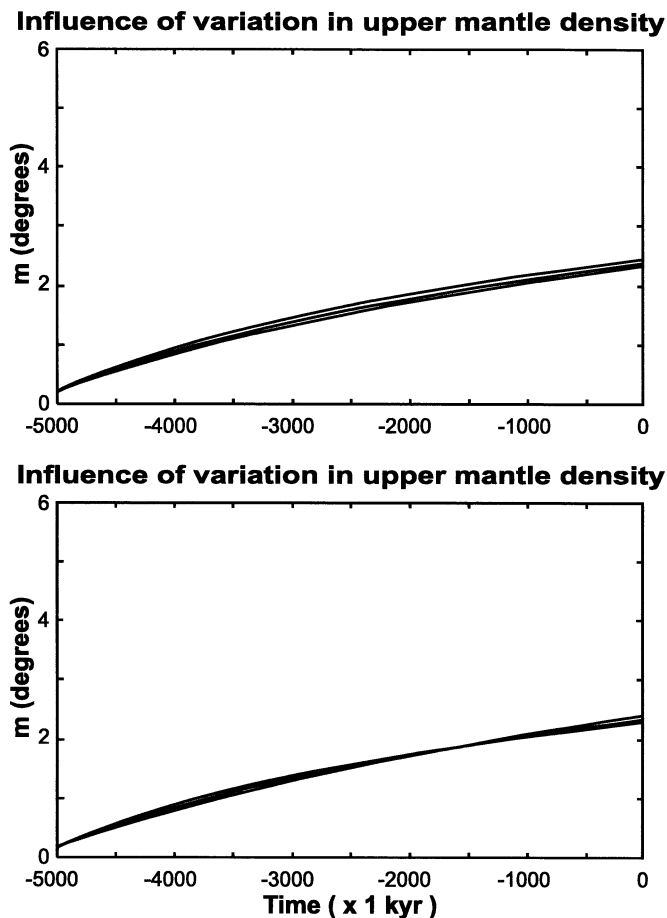


Figure 7.7. Effects of variations in density contrast of the boundaries. The induced polar wander for the Heaviside case in which both load and tidal-effective relaxation are operative is depicted. The viscosity contrast between the transition zone and the shallow upper mantle is 5, between the lower and upper mantle 10. The top panel shows the effects of varying the lithosphere/shallow upper mantle densities. The three curves show the cases for values of 4000, 4100 and 4200 kg/m^3 , with the top curve being the case of the lowest value for the lithosphere/shallow upper mantle density. The bottom panel does the same for variations in the lower mantle density. Values of 4400, 4508 and 4600 kg/m^3 are chosen, with the top curve for the first few million years after root formation onset representing the case of the highest value for the lower mantle density. As is vaguely visible, the order of top and bottom curves changes after some time. Redrawn from Figure 7 in Vermeersen *et al.* (1994).

$$f(t) = \frac{t}{t_0}. \quad (7.9)$$

For the case of the mantle also relaxing to the load we have

$$\mathbf{m}(t) = -i\sigma_r \frac{\mathbf{G}(\sigma, \theta, \phi)}{t_0} \left(\frac{1}{2} A_0^* t^2 - \sum_{i=1}^M \frac{\gamma_i}{a_i} t + \sum_{i=1}^M \frac{\gamma_i}{a_i^2} (e^{a_i t} - 1) \right). \quad (7.10)$$

The results as a function of the viscosity contrasts are shown in Figure 7.8.

The polar wander rates at present with this linear model can be up to about 1 deg/Myr (case of uniform mantle viscosity), which is close to the observed value. For a lower mantle viscosity which is 10 times higher than the value for the upper mantle (Hager, 1991), this rate would be smaller but, for a value of the viscosity of the transition zone equal to that of the upper mantle, the rate of secular drift would still amount to some 0.2 deg/Myr, which is about one-fifth of the value of the contemporaneous rate. The results depicted in the panels of Figure 7.8 strengthens the claim that vertical tectonic movements unrelated to post-glacial rebound can be effective in contributing a considerable amount to present-day polar wander. The linear root formation history, which is active today, is more effective than the Heaviside history in Figures 7.3 and 7.4 because it induces a larger amount of isostatic disequilibrium due to the difficulty mantle modes $M1$ and $M2$ have in keeping up with root growth.

An exponential rate of root formation can be represented by

$$f(t) = \frac{e^{t/t_0} - 1}{e - 1}. \quad (7.11)$$

The uplift is supposed to start at $t = 0$. The normalization factor $e - 1$ makes a function f equal to one after a timespan t_0 has elapsed.

For the case of a mantle relaxing both to the change in load and centrifugal potential

$$\mathbf{m}(t) = -i\sigma_r \mathbf{G}(\sigma, \theta, \phi) \left(-A_0^* t + A_0^* t_0 (e^{t/t_0} - 1) - \sum_{i=1}^M \frac{\gamma_i}{a_i} (e^{a_i t} - 1) + \sum_{i=1}^M \frac{\gamma_i}{1/t_0 - a_i} (e^{t/t_0} - e^{a_i t}) \right). \quad (7.12)$$

Figure 7.9 portrays the induced amount of polar wander as a function of the viscosity stratification of the mantle.

The same tendencies and about the same magnitudes are found in these cases for the present polar wander rates as in the corresponding linear cases in Figure 7.8. The polar wander plots for the three modeled root formation histories have a few characteristics in common. Irrespective of the question whether there is mantle relaxation to the load or not, the polar wander remains very small for the cases in which the viscosity ratio between the lower and upper mantle is large, i.e. for values larger than about 10. The viscosity of the transition zone is not important then. For viscosity ratios between the lower and shallow upper

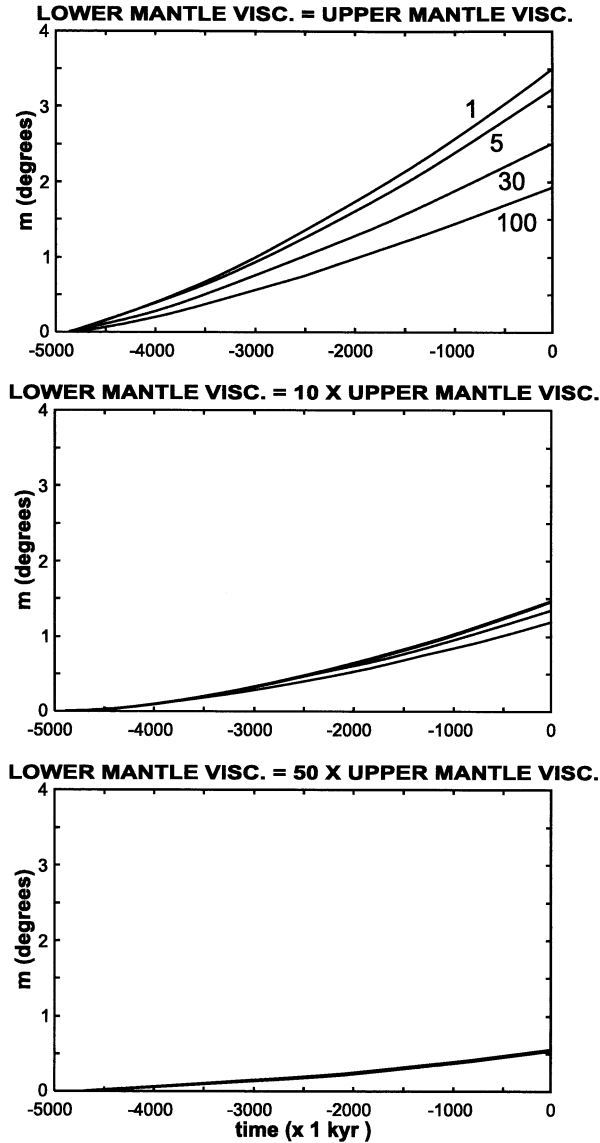


Figure 7.8. Polar wander with full mantle relaxation; linear case. Both load and tidal-effective relaxation are active. The root was formed linearly, starting five million years ago. Redrawn from Figure 8 in Vermeersen *et al.* (1994).

mantle which are smaller than about 10, the induced polar wander is, apart from the time history of the root formation and the amount of mantle relaxation to the load, very sensitive to the viscosity of the transition zone. Given the root formation history and the amount of mantle compensation to the load, the

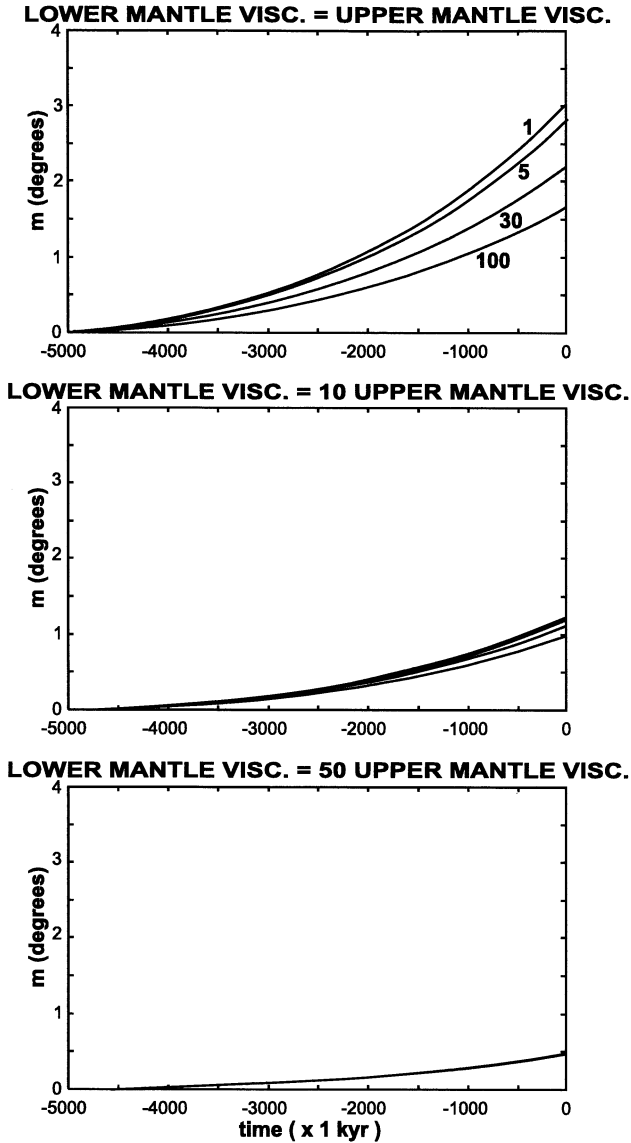


Figure 7.9. Polar wander with full mantle relaxation; exponential case. Both load and tidal-effective mantle relaxation are active. The root is formed exponentially with an exponential rising time of three million years. Redrawn from Figure 9 in Vermeersen *et al.* (1994).

difference in polar wander achieved after 5 million years can be as large as 300 per cent, depending on the transition zone viscosity being between 10^{21} Pa s and 10^{23} Pa s. Most important, however, is that induced rates of polar wander have the right order of magnitude for both the linear and exponential cases. The

most realistic cases (2 per cent of the root associated with permanent deviations from isostasy, viscosity contrasts between the lower and shallow upper mantle of 1 to 10) give substantial contributions to the polar wander signal, though the direction of 82.5° East longitude towards the Himalayas is in the opposite direction of the observed 76° West longitude. We are aware, however, that there are other areas in the world which experience high rates of vertical movement and which are most likely associated with deviations from Airy or Pratt isostasy, even apart from the contributions due to post-glacial rebound, recent ice-water redistributions, subduction and probably events taking place at the core-mantle boundary.

Comparison of the three root formation histories shows that the differences between a linear and exponential evolution are not large. The Heaviside root formation history induces a markedly different polar wander history, but, as noted, this is not a very realistic case. The case of a number of Heaviside root formation histories partitioned over the past few million years would be more realistic. It then becomes important to consider when the changes took place: if root formation happened recently, the small time-scale relaxation modes (if not to the load, then certainly to the centrifugal forcing) are determining the shift of the rotation axis over the globe, just as they do in the post-glacial rebound case. On longer time scales of a few hundreds of thousands of years to a few million years, the effects of all the small time-scale root formation histories tend to result in a linear or exponential uplift history, being strongly dependent on the rheological properties of the transition zone in the case of small viscosity contrasts between the lower and upper mantle. So, the results of the modelings shown in the linear and exponential root formation cases would only give mean values in such a case, around which higher and lower shifts of the rotation axis of shorter duration (on time scales of post-glacial rebound) are likely. The most realistic cases (2 per cent of the root associated with permanent deviations from isostasy, viscosity contrasts between the lower and shallow upper mantle of 1 to 10) are a substantial fraction of the observed polar wander magnitude, though the direction of 82.5° East longitude towards the Himalayas is in the opposite direction of the present-day 76° West longitude.

A few words concerning this direction are in order here. At first it might seem contradictory that the pole is wandering towards the small geoid high of the Himalayas while it is known that due to true polar wander the Earth has the ability to displace geoid heights towards the equator (Ricard *et al.*, 1992). However, one must be aware of the fact that true polar wander is controlled solely by the degree 2 geoid anomaly induced by the source, and for the Himalayas this component cannot be discerned in the global pattern of the Earth. In our modeling, with imposed isostatic disequilibrium or mass deficit at the surface, this component is negative and we can only argue that the degree 2 component used in our calculations is derived from a root excess consistent with the amount

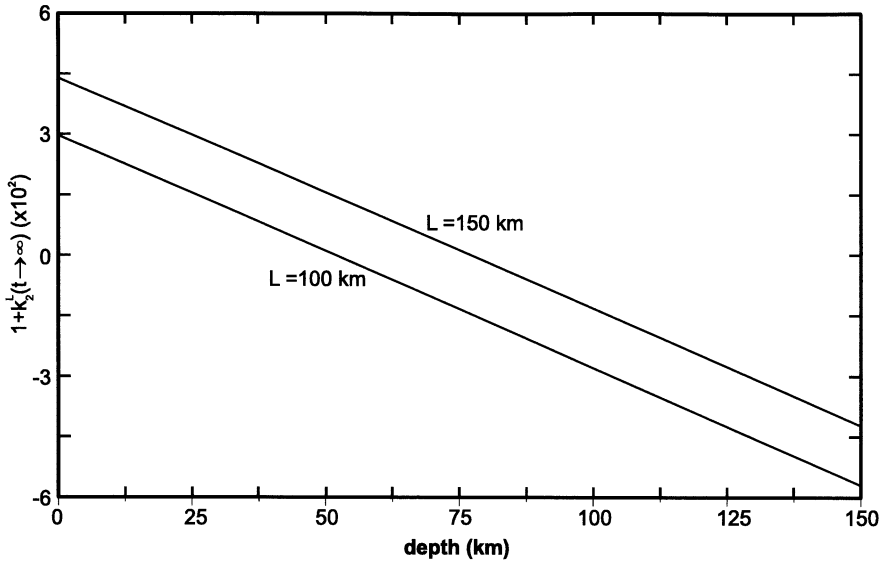


Figure 7.10. Fluid limit $1 + k_2^L(t \rightarrow \infty)$ as a function of the depth of source for lithospheric thicknesses of 100 and 150 km in units of 10^{-2} (Figure 10 in Vermeersen *et al.*, 1994).

of overcompensation inferred from higher harmonics. This shows that the observed high harmonic signal over the Himalayas cannot be used directly to constrain the direction of true polar wander but only to estimate the deviation from isostasy.

We can now focus on the effects of the depth of the mass deficit for the cases of full mantle relaxation to the load. In previous calculations, forcing has been applied at the Earth's surface. This makes sense, as the crust rests upon the lithosphere. In this case, the amount of isostatic disequilibrium maintained by elastic stresses in the 100-km thick lithosphere is close to the 2 per cent inferred from high harmonic geoid anomalies. If the mass deficit is embedded at different depths in the elastic lithosphere, the function $1 + k_2^L(t \rightarrow \infty)$ varies as shown in Figure 7.10 for thicknesses of 100 km and 150 km, respectively.

This function enters linearly the velocity of polar wander after relaxation of the mantle modes, as in equation (7.9) or (7.11), so this velocity for varying depth of the mass anomaly can easily be inferred from Figure 7.10 by linear scaling. It is apparent that $1 + k_2^L(t \rightarrow \infty)$ decays in a linear fashion within the lithosphere, or the first 50 km of the upper mantle for the thinner lithosphere, and changes sign after crossing zero in proximity to the middle of the elastic layer. For a lithosphere of 150 km and a mass anomaly embedded at 50 km, polar wander velocity is decreased, with respect to the uniform viscosity model in Figure 7.9, by 30 per cent, or by a factor 15 for a depth of 75 km in proximity

of the zero of $1 + k_2^L(t \rightarrow \infty)$, corresponding to a velocity of 0.07 degrees per million year, less than 10 per cent of the observed value. Note the change in sign of this function in proximity to 50 and 75 km, respectively, which is responsible for the direction of the polar wander away from the Himalayas, opposite to previous results. The behavior of the function $1 + k_2^L(t \rightarrow \infty)$ is due to the flexural properties of the lithosphere, also noted in Ricard *et al.* (1992); it is remarkable that it vanishes in the middle of the lithosphere in correspondence to the neutral plane of the elastic layer, where horizontal stresses change sign. This finding indicates that the degree 2 geoid anomaly and hence polar wander, for a source embedded in the elastic top layer, are maintained by the elastic stresses stored in the lithospheric upper mantle. In short, for a deep source, elastic stresses are unable to sustain the mass anomaly and the geoid signal due to viscous flow in the upper mantle. For models allowing full relaxation in the mantle, polar wander velocities due to mountain building are extremely sensitive, both in amplitude and direction, to the proximity to the overcompensated roots. For deep sources in proximity of the neutral plane of the elastic lithosphere, polar wander is negligible; mountain building may be effective in driving polar wander only if there is a tectonic mechanism which maintains a permanent deviation from isostasy. In general, the best way to model crustal deformation in which the stresses in the lithosphere do not maintain a permanent deviation from isostasy is to make use of models in which the lithosphere is absent (lithospheric thickness is zero), corresponding to the case of $1 + k_2^L(t \rightarrow \infty) = 0$ in Figure 7.10. Vice versa, deviations from isostasy maintained by compressive lithospheric deviatoric stresses are well modeled by the presence of an elastic layer.

The relation between non-glacially induced tectonic movements and changes in rotation is a difficult one to establish. Apart from the uncertainties in the rheological mantle stratification of our planet, which is assumed to be the major unknown in post-glacial rebound modeling, the vertical movement and erosion rates of various regions, and the deviations from isostasy maintained by compressive intraplate stresses and (continental) lithospheric rheologies, present extra uncertainties in establishing the relation.

Given the restrictions of our approach, the best we can do concerning load relaxation is to assume that a certain amount of the formed low density root is unrelaxed by mantle relaxation. The simulations then show a polar wander which, for reasonable root formation rates and mantle stratifications, can significantly contribute to true polar wander, although we expect that deep-seated density anomalies, such as those at subduction zones, are more efficient in triggering considerable excursions of the axis of rotation because of the large amount of flow induced in the mantle. For a lower to shallow upper mantle viscosity contrast which is not too large (i.e., smaller than about 10), the induced polar wander is sensitive to the viscosity value of the transition zone.

Finally, in this section we only considered one region which experiences uplift under non-isostatic conditions, though this region is a very active and, potentially, the most important one. It was not our aim in this section to give an exhaustive treatment of all the possible regions in the world which undergo active lithospheric vertical movements accompanied by deviations from isostasy. Rather, we wanted to present a case study of only one. The limits of our model are the simplified rheology in proximity to the surface, the lack of lateral-thickness variations and absence of faults, which only allows for an order of magnitude estimate of true polar wander induced by mountain building.

2. SUBDUCTION EFFECTS ON TPW

2.1 NON-LINEAR ROTATION THEORY

When dealing with marked excursions of the rotation axis, such as TPW driven by subduction, we must use, as shown in this section 7.2 reworked from Ricard *et al.* (1993), the fully non-linear equation (3.1), where the inertia tensor $\mathbf{I}^*$ can be divided into three contributions of decreasing amplitudes

$$\mathbf{I}^* = I\delta_{ij} + \frac{k_2(t)a^5}{3G} * (\omega_i\omega_j - \frac{1}{3}\omega^2\delta_{ij}) + (\delta(t) + k_2^L(t)) * \Delta I_{ij}(t), \quad (7.13)$$

where the asterisk in the right member denotes the time convolution. The three terms on the right-hand side are of the order of $0.33 M_E a^2$, $1.08 \times 10^{-3} M_E a^2$ and $10^{-5} M_E a^2$, where M_E and a are the mass and the radius of Earth.

The first term $I\delta_{ij}$ denotes the tensor of a spherical non-rotating Earth and is equal to $0.33 M_E a^2$. The second term results from the centrifugal potential deforming the Earth. Given that the trace of the inertia tensor is conserved during the deformation ($I_{ii} = 0$), we can use equations (3.31) and (3.33) to show that this potential is proportional to $\omega_i\omega_j - (1/3)\omega^2\delta_{ij}$. This means that any change in rotation is equivalent to a new potential applied to the Earth's surface. When under such a boundary condition, the planet matches an inertia tensor equal to the convolution of $k_2(t)$, the tidal Love number of harmonic degree 2, with the time history of the centrifugal potential (Takeuchi *et al.*, 1962). The contribution related to the rotation is thus

$$\frac{k_2(t)a^5}{3G} * (\omega_i\omega_j - \frac{1}{3}\omega^2\delta_{ij}). \quad (7.14)$$

Equation (7.14) shows that a planet rotating at a constant velocity Ω reaches a steady-state axi-symmetrical shape with a polar inertia

$$C = I + \frac{2k_f a^5}{9G} \Omega^2 \quad (7.15)$$

and two equatorial inertia

$$A = I - \frac{k_f a^5}{9G} \Omega^2, \quad (7.16)$$

where the fluid tidal Love number k_f is the limit of $k_2(t)$ for very large times, given by equation (3.37). The inertia I thus equals $(C + 2A)/3$ and the amplitude of the rotational contribution to inertia is

$$C - A = \frac{k_f a^5}{3G} \Omega^2. \quad (7.17)$$

If we assume that the Earth is close to hydrostatic equilibrium, we find this value can be identified with the observed flattening $C - A = 1.08 \times 10^{-3} \text{ Ma}^2$.

The third and last term in the inertia tensor $\mathbf{I}^*$ is related to the mass redistribution due to geophysical processes inside the Earth or at its surface, with k^L denoting the loading Love number appropriate for surface loading, section 1.8.2, or internal load, section 1.8.1. We can write this term as

$$(\delta(t) + k_2^L(t)) * \Delta I_{ij}(t), \quad (7.18)$$

where $\Delta I_{ij}(t)$ represents the inertia changes due to a given geophysical process, without taking into account any dynamic deformation. These inertia changes act directly on the planet through the Dirac delta function and $\Delta I_{ij}(t)$ generalizes, for surface loads, equations (3.58) and (3.59), where instead of having ΔI_{13}^R , ΔI_{23}^R we now deal with the complete inertia perturbation tensor. The convolution accounts for the effects of isostatic compensation. The amplitude of ΔI_{ij} can be estimated for the excitation sources. For example, the Pleistocene deglaciation in Chapter 4 corresponds to a change of inertia with an amplitude of about 10^{-5} Ma^2 .

In Chapters 3 and 4, or in the previous section on mountain building, we made use of linearized rotation equations, so that only the perturbations of ω with respect to a starting vector Ω were considered. The general non-linear problem governed by the fully non-linear rotational equations has been solved for very simple models and for constant excitation sources (Munk and MacDonald, 1960; Lefftz *et al.*, 1991). This section deals with the effects of a time-dependent mantle mass redistribution on the Earth's rotation. The impact of slow varying processes on rotation was first considered within a self-consistent fully non-linear approach by Ricard *et al.* (1993). Internal mass redistribution could induce a TPW velocity comparable to the observed value of around 1 Deg/Myr and eventually move the Earth's pole to a great extent. The practical problem in dealing with these issues is to compute realistic Love numbers and to efficiently perform the convolutions in the previous equations.

When dealing with large excursions of the rotation axis, it is necessary to overcome the numerical difficulties of solving the non-linear problem arising from the interaction of different time scales spanning various decades. These

Table 7.1. Parameters for the 4-layer fixed-boundary contrast Earth model. r is the distance with respect to the center of the Earth, ρ the density of the layer, and μ the rigidity.

layer	r (km)	ρ (kg/m ³)	μ (N/m ²)	
1	6371 – 6271	2689	2.82×10^{10}	lithosphere
2	6271 – 5951	4430	8.37×10^{10}	upper mantle
3	5701 – 3480	4919	2.17×10^{11}	lower mantle
4	3480 – 0	10927	0	inviscid fluid core

time constants range from zero (the elastic impulse) to a relaxation time of several million years associated with the $M1$ mode, the slowest relaxation mode in the four-layer earth model considered in this chapter, Table 7.1.

That a very large time span also holds for the geophysical excitations prevents any direct approach to the integro-differential system above. The complexity of numerical integration is further increased by the fact that $\omega(t)$ does not monotonously vary but revolves at a Chandler period of about 435 days around an average solution. An approximate solution of these equations is thus necessary. Since this section deals with very slow geophysical processes like plate motions and mantle convection, our excitation function varies with time constants larger than 1 Myr. To begin with, let us assume that our earth model has no internal non-adiabatic density discontinuity, so that the $M1$ mode is not excited and the slowest relaxation time is much smaller than 1 Myr (see section 1.6). In this case, $|s_i t_i| \gg 1$ or $|s| \ll s_i$ and we can thus approximate the loading Love number as

$$k_2^L(t = \infty) = k_e^L - \sum_{i=1}^M \frac{k_i^L}{s_i}. \quad (7.19)$$

This enables us to identify the time-dependent isostatic Love number $k_2^L(t)$ with its fluid limit k_f^L .

We cannot directly make use of the fluid limit k_f for the tidal case because the time-dependent readjustment of the equatorial bulge during polar wander must be self-consistently taken into account. Since the approximation is somewhat more complex here, we must allow for the possibility of a time-dependent readjustment of the equatorial bulge during polar wander. Thus, from

$$k_2(s) = k_e + \sum_{i=1}^M \frac{k_i}{s - s_i} \quad (7.20)$$

and $|s| \ll s_i$ to first order in s/s_i , we must use expansion (3.83) appropriate for the long-term behavior of the rotation equation to obtain

$$k_2(s) = k_e - \sum_{i=1}^M \frac{k_i}{s_i} \left(1 + \frac{s}{s_i}\right). \quad (7.21)$$

By defining the fluid tidal Love number as in equation (3.84), we have

$$k_2(s) = k_f(1 - T_1 s), \quad (7.22)$$

where T_1 is defined as in equation (3.86).

Expansion (7.22) is valid for $|s| \ll |s_{M1}|$, meaning that our approximation holds for geophysical processes slower than the relaxation time controlling the readjustment of the non-adiabatic discontinuity at the 670 km depth. The quality of the approximation for the tidal Love number is reinforced by the fact that, for most of the realistic models of the Earth's mantle, the amplitude of the residue k_{M1} is negligible. Put another way, this vanishing strength means that a change in the Earth's centrifugal potential induces no appreciable displacement of the 670-km depth interface.

$M1$ plays a major role in the loading problem. When $k^L(s)$ is computed for masses found inside the mantle, the amplitude k_{M1}^L associated with the $M1$ mode can be large. We thus separate the slowest mode from the fast relaxing ones and use the following equation for $k^L(t)$

$$k_2^L(t) = k_f^L + \frac{k_{M1}^L}{s_{M1}} e^{s_{M1}t}. \quad (7.23)$$

In this equation, expressing the convolution of the loading Love number with a Heaviside function, it is assumed that on the given time scale all the modes, except possibly the $M1$ mode, have already relaxed and the convolution (7.18) between the exponential part of $k^L(t)$ and ΔI_{ij} can thus be easily performed numerically.

The quantity $1 + k_f^L$ as a function of depth of the excitation source is shown in Figure 7.11 for a number of earth models.

These functions are zero for mass anomalies close to the core-mantle boundary. For non-adiabatically stratified models (bottom row), the mass anomalies are also perfectly compensated for when they lie close to the density jump at 670-km depth. For masses at the surface, $1 + k_f^L$ is also zero in the case without the lithosphere (left column), while a slight undercompensation is supported by elastic stresses when the lithosphere is present (right column). In the case of moderate viscosity increases in the mantle, a positive mass anomaly gives rise to a negative $1 + k_f^L$; larger viscosity variations can impose a positive $1 + k_f^L$. As the inertia tensor is related to the geoid of degree 2, the excitation functions $1 + k_f^L$ only differ from the geoid kernels by a normalization factor (Ricard *et al.*, 1984; Richards and Hager, 1984). The long-term limits of the viscoelastic models without purely elastic lithospheres exactly reproduce what has been found for purely viscous steady-state models.

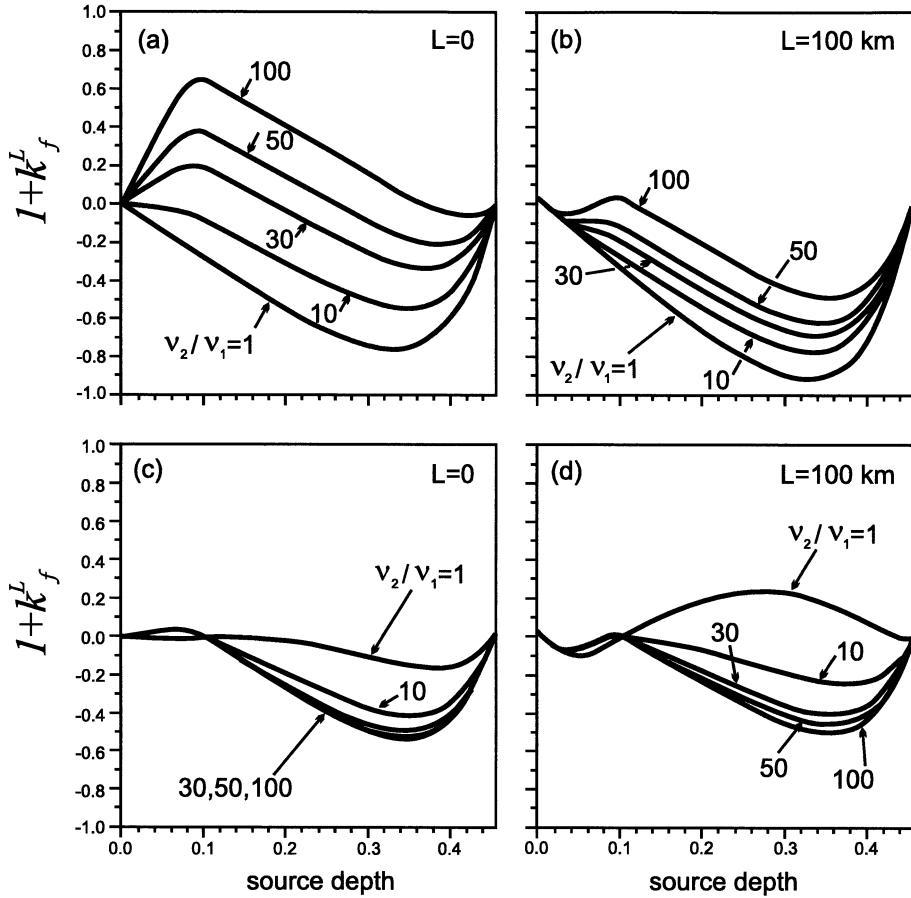


Figure 7.11. Perturbation in the inertia in the long-term approximation due to an inertia source of unit amplitude introduced at time $t = 0$ in the mantle. This perturbation $1 + k_f^L$ is plotted as a function of the mass anomaly depth for a viscosity jump at the upper-lower mantle interface by a factor of 1, 10, 30, 50 or 100. In the first row, the mantle density is uniform, in the second row it increases by 9 per cent at 670 km depth. A lithosphere is present in the right column. Redrawn from Figure 4 in Ricard *et al.* (1993).

The equivalence between long-term viscoelastic models and viscous steady-state models needs some clarification. That is, when $t \rightarrow \infty$ or $s \rightarrow 0$, our viscoelastic model assumes the character of a Newtonian viscous fluid (Wu and Peltier, 1982) and has the same surface Love numbers. For internal loads, the displacements inside the viscoelastic mantle tend to infinity when $s \rightarrow 0$, but the associated velocities approach finite values equal to those obtained via steady-state viscous models. When $s \rightarrow 0$, the viscous limit of a viscoelastic model cannot be directly obtained by setting $s = 0$ in the rheological law,

which would correspond to the rheology of an inviscid fluid. That steady-state viscous models can be used even for time-dependent loads is due to the fact that interface deformations are generally faster than the time-varying positions of the loads. However, the expression of $k_2^L(t)$, which explicitly includes the time dependence of the $M1$ mode, allows us to take into account the upper-lower mantle interface, which readjusts slowly. The need for keeping the $M1$ mode in the convolution of the load history with the isostatic Love numbers is illustrated in Figure 7.12. We computed the time-dependent excitation function $1 + k_f^L$ for four chemically stratified models. While in the top row of Figure 7.12 the mantle viscosity is uniform, it increases by a factor of 50 in the lower row. The right column differs from the left by the presence of a lithosphere. The Love numbers deduced are plotted by a dashed line, 0.2 Myr (panel a) or 1 Myr (panels b, c and d) after the imposition of a Heaviside load. Since at such time the Love numbers are far from reaching their asymptotic values (full lines), it is necessary to account for the explicit time-dependence of the $M1$ mode when chemically stratified models are used.

Inserting equation (7.22) into (7.13) and taking into account that the image of a time derivative $\dot{f}(t)$ is $sf(s)$ in the s -space, we can express the total inertia tensor as

$$\mathbf{I}^* = I\delta_{ij} + \frac{k_f a^5}{3G}(\omega_i \omega_j - \frac{1}{3}\omega^2 \delta_{ij}) - \frac{k_f T_1 a^5}{3G}(\dot{\omega}_i \omega_j + \omega_i \dot{\omega}_j - \frac{2}{3}\omega_l \dot{\omega}_l \delta_{ij}) + E_{ij}, \quad (7.24)$$

where

$$E_{ij} = (\delta(t) + k^L(t)) * \Delta I_{ij}(t). \quad (7.25)$$

as in equation (7.13).

By neglecting the terms $\ddot{\omega}$ and $\dot{\omega}^2$ for consistency with the previous approximations, these definitions and the basic equation (3.4) give us (Ricard *et al.*, 1993):

$$\Delta_{ij}(\omega) \dot{\omega}_j + \Gamma_{ij}(\omega, E, \dot{E}) \omega_j = 0, \quad (7.26)$$

where

$$\Delta = \begin{pmatrix} I & \frac{k_f T_1 a^5}{3G} \omega^2 \omega_3 & -\frac{k_f T_1 a^5}{3G} \omega^2 \omega_2 \\ -\frac{k_f T_1 a^5}{3G} \omega^2 \omega_3 & I & \frac{k_f T_1 a^5}{3G} \omega^2 \omega_1 \\ \frac{k_f T_1 a^5}{3G} \omega^2 \omega_2 & -\frac{k_f T_1 a^5}{3G} \omega^2 \omega_1 & I \end{pmatrix} \quad (7.27)$$

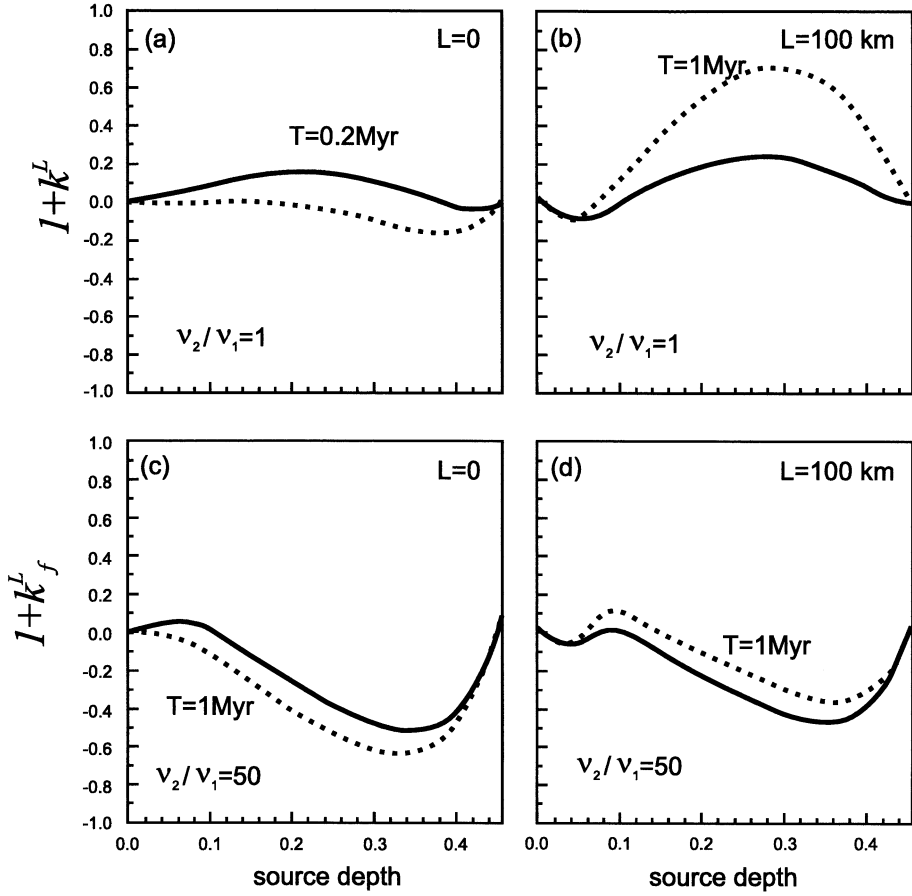


Figure 7.12. Inertia perturbation $1 + k_f^L$ computed at time $t = 0.2\text{ Myr}$ or $t = 1\text{ Ma}$ after the introduction of a mass anomaly (dashed line). A solid line depicts the fluid limit obtained at very long times. The models used in the four panels are stratified in density and the long-term behavior is controlled by the $M1$ mode. On top, the mantle viscosity is uniform, at bottom it increases by a factor of 50. A lithosphere is present in the right column. For cases (b), (c) and (d) the asymptotic regime is only reached after millions of years. Redrawn from Figure 5 in Ricard *et al.* (1993).

$$\Gamma = \begin{pmatrix} \dot{E}_{11} & \Sigma_3 & -\Sigma_2 \\ -\Sigma_3 & \dot{E}_{22} & \Sigma_1 \\ \Sigma_2 & -\Sigma_1 & \dot{E}_{33} \end{pmatrix} \quad (7.28)$$

and

$$\Sigma_i = E_{ij}\omega_j. \quad (7.29)$$

Equations (7.25)-(7.29) can also be found in Ricard *et al.* (1993). Since the diagonal terms of Δ are smaller than the non-diagonal terms, a further approximation might neglect them (Lefftz *al.*, 1991). In such a case, Δ is not invertible and the conservation of ω^2 must be imposed. Even by including its diagonal terms, the matrix Γ is not numerically ill-conditioned. The approximation, i.e. neglecting the diagonal terms, provides insight into the basic physics of polar wander on a long-term scale. This approximation is equivalent to replacing equation (3.4) by

$$\mathbf{I}^* \cdot \omega = \alpha\omega, \quad (7.30)$$

where α is unknown. The Euler equation degenerates into an eigenvalue problem: the angular momentum $\mathbf{I}^* \cdot \omega$ remains constantly parallel to the angular velocity ω . Likewise, the equation above indicates that the non-diagonal terms of $\mathbf{I}^*$, namely ΔI_{13} and ΔI_{23} , in a reference frame where z coincides with ω , are zero in the long-term approximation. Note, of course, that the inertia matrix $\mathbf{I}^*$ includes not only the effects of internal masses in a dynamic non-rotating Earth but also the rotational deformations. The main inertia of the exciting source is not necessarily parallel to the rotation axis. We can check the validity of our equations by computing the change in the rotation vector $(0,0,\Omega)$ when perturbed at time $t = 0$ by a change in inertia. By taking into account that (equation (7.17))

$$C - A = \frac{k_f a^5}{3G} \Omega^2, \quad (7.31)$$

by disregarding the terms that include ω_1 and ω_2 , since we start from an initial configuration $(0,0,\Omega)$ in which these are zero, and by assuming that ω_3 is equal to Ω , the previous matrix equation (7.26) yields

$$\dot{\omega}_1 + i\dot{\omega}_2 = \Omega \left(\frac{1}{T_1 + iI/[\Omega(C - A)]} \right) \frac{E_{13} + iE_{23}}{C - A} \quad (7.32)$$

$$\dot{\omega}_3 = -\Omega \frac{\dot{E}_{33}}{I}. \quad (7.33)$$

Polar motion and change in length of day are governed by these equations, decoupled as in equations (3.16) - (3.18). The first equation above, appropriate for long time scales, coincides with equations (3.16), (3.17) and (3.18) once the explicit expression of T_1 given by (3.86) is considered, except for the appearance of the moment of inertia I , instead of the equatorial moment of inertia A , due to our approximations.

This equation affords a better understanding of the differences between the inertia tensor of the non-rotating dynamic Earth E_{ij} and the rotating dynamic Earth $\mathbf{I}^*_{ij}$. The observed present-day polar wander has a velocity of 1 Deg/Myr, is directed toward Hudson Bay and is mainly related to post-glacial rebound (Vermeersen *et al.*, 1997). Although the elapsed time since the deglaciation is too short for a complete relaxation of the modes involved in the tidal relaxation process, we can use this equation to estimate the order of magnitude of the excitation presently driving the pole. For $T_1 = 20$ Kyr, which corresponds to a viscosity increase in the mantle of around 25, as shown in Figure 3.5, the excitation $E/(C - A)$ amounts to $3.5 \cdot 10^{-4}$, a value which can be translated into terms of a geoid anomaly of degree 2 and order 1 of about 7 m. While the tensor E of the non-rotating Earth presents non-diagonal terms E_{31} and E_{32} associated with a 7-m high geoid anomaly of degree 2 and order 1, the tensor $\mathbf{I}^*$ of the rotating Earth is purely axi-symmetrical and associated with a geoid without terms of degree 2 and order 1. The very existence of tensor E only arises from the observation of polar wander. We can see from this linearized equation that the planet, subjected to a change of inertia of order E , will wander with a characteristic time of order $T_1(C - A)/E$. A lower bound for this time scale can be estimated from the ratio of the Earth's flattening ($=21$ km) to the geoid height anomalies of degree 2 ($=100$ m). This characteristic time is thus larger than $200 T_1$. That is, under a change of inertia, the Earth can shift its rotation pole from a starting position to a new position on a time scale larger than a few 100 kyr or 1 Myr, depending upon its internal stratification.

Resolving the non-linear equations above can be performed by a standard Runge-Kutta algorithm, as done in Ricard *et al.* (1993), which generalizes for a stratified Earth and a complex load history, the solution obtained by Milankovitch (1934) in a discussion of possible polar wander induced by the distribution of continents on top of a homogeneous viscoelastic Earth.

2.2 POLAR WANDER VELOCITY FOR A DISTRIBUTION OF SLABS

An idealized model for mass redistribution within the mantle, first modeled in Ricard *et al.* (1993), is now presented to show the effects of convection on the Earth's rotation. This simple model catches some of the basic features of mantle convection and is inspired by Goldreich and Toomre (1969). We assume that slabs randomly distributed on the sphere, the analogues of Gold's (1955) beetles, are falling inside the mantle with a new slab sinking every two million years. Their velocity in the upper mantle is equal to 10 cm/yr and is reduced according to the increase in mantle viscosity in the lower mantle. On average, eight slabs are present at the same time in the upper mantle and all have the same mass of 2×10^{19} kg. We computed from this naive mantle convection model, the inertia tensor of our model Earth in a constant geographical framework for

a time span of 500 Myr. The results are depicted in the following Figure 7.13, panels (a) and (b), for two different rheological stratifications. Our earth model is adiabatically stratified in both cases. In panel (a), the mantle is uniform, whereas in panel (b) the lower mantle viscosity is increased by a factor 30. The bottom part of each panel shows the variations of the three components of the rotation vector in a geographical framework.

The amplitude of the initial rotation vector has been used to normalize the velocity components. The top part of each panel shows the rate of TPW in Deg/Myr; the dashed line represents the average TPW velocity. Note the striking differences in the rotation behavior between a uniform and a stratified mantle, panels (a) and (b). A viscosity increase at 670 km depth inhibits mantle flow and slows the average TPW velocity below 1 Deg/Myr. This slow-down is due to three concurrent effects. First, the viscosity jump decreases the slab velocities and, hence, the amplitudes of the time derivatives of the inertia tensor. Second, it decreases the amplitude of the loading excitation term $1 + k_f^I$, at least for a viscosity increase lower than 100. And, third, it increases time T_1 of the rotational response by channeling the flow into the upper mantle. Each slab in the model with a homogeneous mantle crosses the whole mantle in about 30 Myr, so that, after a transient regime of 30 Myr, our model reaches a steady-state behavior where the slabs are uniformly distributed in the mantle.

3. POLAR WANDER ON THE EARTH, MOON, MARS AND VENUS

Despite the uncertainties due to our lack of information concerning mantle rheology, for most of the solid planets we can deduce reasonable bounds for the rotational number R_o defined in equation (3.91) that are sufficient to describe the basic characteristics of their long-term rotation, as shown in this section 7.3 reworked from Spada *al.* (1996). Our results are summarized in Figure 7.14, where R_o is estimated via equation (3.91) and plotted as a function of T_1 for various models of planetary interiors. For the models of the planetary bodies considered, Figure 7.14 shows the parameters characterizing their internal structure and rotation.

The first earth model, Ear-1 (solid squares as all the other Earth's models), is characterized by a uniform mantle with viscosity $\nu = 10^{21}$ Pa s overlaid by a 100-km-thick elastic lithosphere; model Ear-2 includes, in addition, a density increase across the 670-km-depth seismic discontinuity, with $\Delta\rho/\rho = 9$ per cent (Sabadini and Yuen, 1989). Numerical values of densities and elastic rigidities needed for the derivation of T_1 are the same as those employed in previous studies (Spada *et al.*, 1993).

Observed values for the dynamic form factor, normalized mean inertia, and mean rotational velocity of the Earth are $J_2 = 1.08263 \times 10^{-3}$, $I^* = 0.3306$ and $\Omega = 7.292115 \times 10^{-5} \text{ s}^{-1}$. Values of R_o and T_1 for the Earth register their

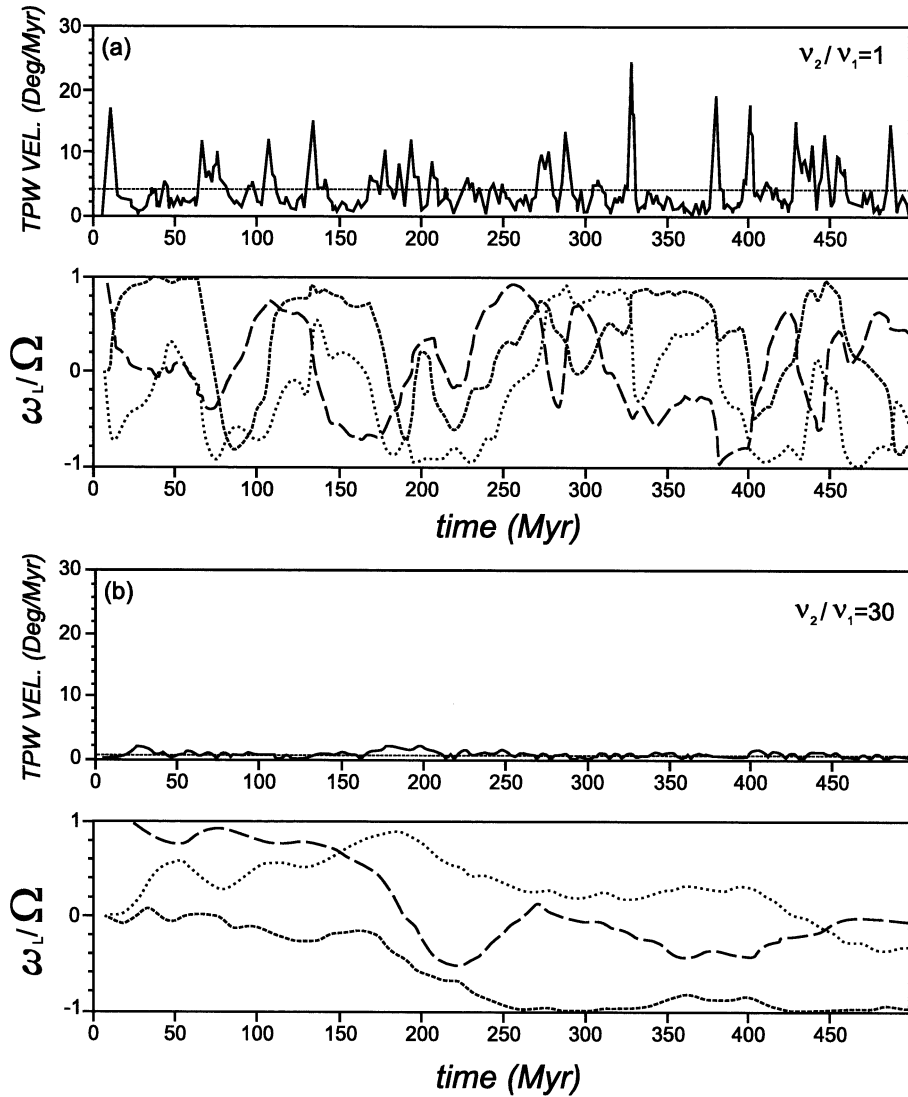


Figure 7.13. Velocity of polar wander and normalized components of the angular velocity vector in an Earth-fixed reference frame induced by a random distribution of sinking slabs. In panel (a) the mantle is uniform and the Earth rotation axis is highly unstable. Increasing viscosity by a factor 30 in the lower mantle (b) drastically reduces the amplitude of TPW rate. The 670 km depth interface is supposed to be a phase-change discontinuity. Redrawn from Figure 9 in Ricard *et al.* (1993).

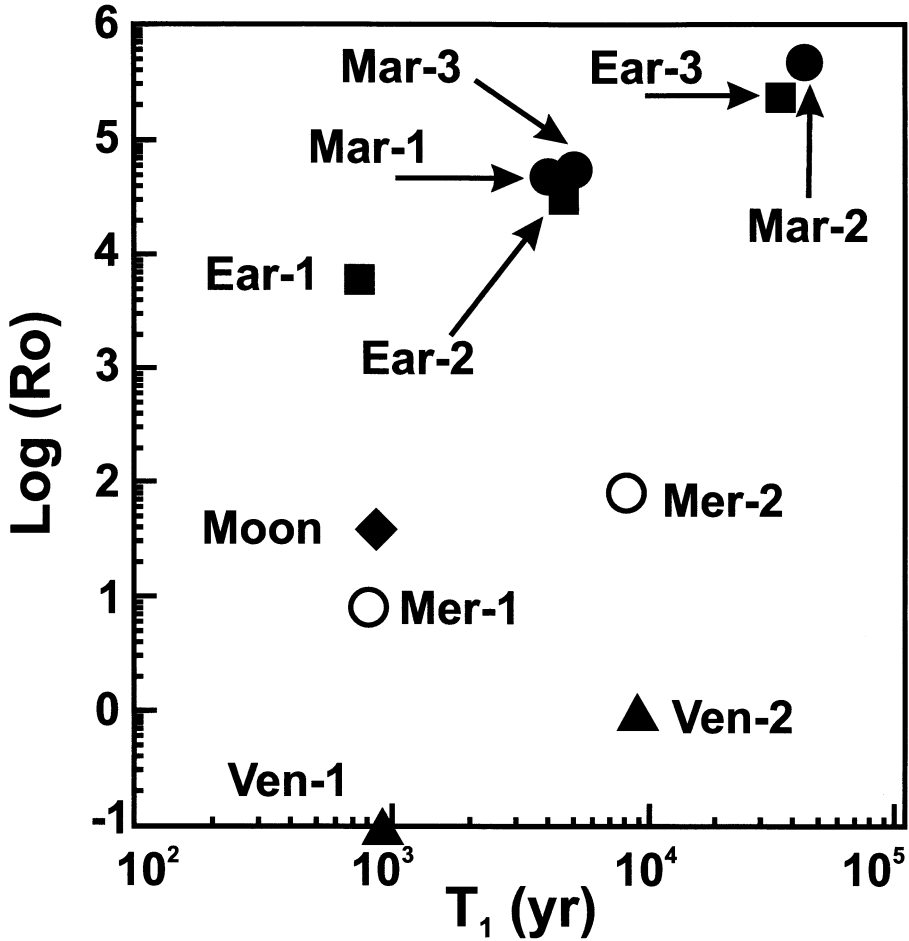


Figure 7.14. Diagram showing the values assumed by the rotational number R_o as a function of the time-scale T_1 for various models of planetary interiors. (Figure 1 in Spada *et al.*, 1996).

largest amplitudes for model Ear-3, characterized by the same density profile as Ear-2 but differing by a viscosity jump of a factor of 50 at 670-km depth. Values of R_o for the Earth vary between $\sim 6 \times 10^3$ for model Ear-1 and 2.5×10^5 for Ear-3; intermediate values of R_o ($\sim 2 \times 10^4$) are found for model Ear-2.

For Mars (solid circles in Figure 7.14) we have employed two different values for mantle viscosity: $\nu = 10^{21}$ Pa s for both models Mar-1 and Mar-3 and $\nu = 10^{22}$ Pa s for Mar-2. These values were chosen on analogy with the Earth because of the large uncertainties in the rheological profile of Mars (Phillips and Lambeck, 1980). Lithospheric thickness is fixed at 300 km for Mar-1 and Mar-2 and at 100 km for Mar-3 (Phillips and Lambeck, 1980). The core radius for the

whole set of Martian models is 1300 km with $J_2 = 1.960 \times 10^{-3}$, $I^* = 0.345$ and $\Omega = 7.09 \times 10^{-5} \text{ s}^{-1}$ (see Balmino *et al.*, 1982; Bills, 1989; Cole, 1978). The Martian density profile is consistent with the results reported by Johnston *et al.* (1974). The small difference between the R_o values in models Mar-1 and Mar-3 indicates that the lithospheric thickness has a minor influence on the rotational number. While the R_o values for Mar-1 and Mar-3 are very close to those estimated for Ear-2, R_o for Mar-2 attains a value slightly exceeding Ear-3.

The two different models considered for Venus (solid triangles in Figure 7.14) are characterized by a 100-km-thick elastic lithosphere and by a core radius of 3240 km. The reference values for mantle viscosity of Venus are $\nu = 10^{21}$ Pa s for model Ven-1 and 10^{22} Pa s for Ven-2 and are consistent with those suggested by the observed offset between its spin axis and the direction of largest inertia (Mottinger *et al.*, 1985; Nerem *et al.*, 1993; Konopliv and Sjogren, 1994). A commonly adopted value for the normalized mean moment of inertia of Venus is $I^* = 0.33$ (Mottinger *et al.*, 1985). The slow rate of rotation ($\Omega = 2.9910^{-7} \text{ s}^{-1}$) and small dynamic form factor ($J_2 = 4.0 \times 10^{-4}$) are responsible for the low R_o values found for Venus ($R_o \sim 10^{-1}$). Rough estimates of R_o and T_1 for the Moon and Mercury are also shown. For the Moon (diamond) they are based on classical results about the internal constitution and rheology of this body (Toksöz, 1974; Nakamura, 1983). Adopted numerical values of J_2 and I^* for the Moon are 2.036×10^{-4} and 0.3905 (Bills and Ferrari, 1977, 1980). The two considered models for Mercury (open circles in Figure 7.14) are characterized by a 100-km-thick lithosphere and by a silicate mantle with viscosity of 10^{21} Pa s (Mer-1) and 10^{22} Pa s (Mer-2). The radius of the iron core is 1700 km (~ 0.8 per cent of the radius of the body), and mantle density is fixed at 3000 kg/m^3 (Cole, 1978). For this model, $I^* \sim 0.337$ and we have adopted the reference values $J_2 = 80 \times 10^{-6}$ and $\Omega = 1.219 \times 10^{-6} \text{ s}^{-1}$ (Zharkov and Trubystin, 1978). The most striking feature in Figure 7.14 is that the rotational number R_o for models Ven-1 and Ven-2 ranges between $\sim 10^{-1}$ and 1, i.e. 4 to 6 orders of magnitude smaller than the corresponding values for the Earth and Mars. The smallness of R_o for Venus in comparison with Mars and the Earth is primarily a consequence of the small rate of rotation and the resulting small dynamical form factor J_2 (Mottinger *et al.*, 1985), if we assume that its gross viscosity structure is similar to that of the other two planets (Phillips and Lambeck, 1980; Kiefer *et al.*, 1986).

Venus would only attain a rotational number comparable with that of the Earth and Mars if its viscosity were several orders of magnitude higher than the Earth's viscosity, which is unlikely due to the convective properties of this planet as revealed by its gravity field (Kiefer *et al.*, 1986). Mercury and the Moon are characterized by R_o values ranging between 10 and 10^2 .

We can discuss the behavior of the Liouville equations in the two limits, $R_o \ll 1$ and $R_o \gg 1$. Figure 7.15 deals with the case of a point mass δm acting within the mantle of the planet and located in the yz -plane of the non-inertial Cartesian frame rotating with the body. For $R_o \gg 1$ and constant inertia perturbation ΔI_{23} , equation (3.90) provides $m_x \sim 0$ and

$$m_y = \frac{\Delta I_{23}}{(C - A) T_1} t, \quad (7.34)$$

which shows that the body's pole of rotation moves in the yz -plane, toward δm if ΔI_{23} is positive and in the opposite direction if ΔI_{23} is negative. This is expected for the Earth and Mars, whose R_o exceeds 10^4 (see Figure 7.14). The linearized Liouville equations (3.90) for $R_o \ll 1$ predict a different pole path,

$$m_x = \frac{\Omega \Delta I_{23}}{A} t, \quad (7.35)$$

which shows that the pole initially moves at right angles with respect to the plane containing the mass anomaly. For Venus, whose rotation number R_o lies in the range $10^{-1} - 1$, the pole wanders in a direction which forms an angle $\gamma = \tan^{-1} R_o$ with respect to the xz -plane.

Both equations (7.34) and (7.35) are valid for small excursions of the axis of rotation. Numerical analyses of the fully non-linear Liouville equations for both Mars and Venus, in Spada *et al.* (1996a) show that the qualitative behavior of these asymptotic expressions are preserved in the long-term limit. Note that equation (7.35) provides only the initial amplitude and direction of the damped wobble which is retrieved from the non-linear Liouville equations. Figure 7.15 helps to explain the physical meaning of the rotational number R_o : it describes how the pole of rotation approaches the mass anomaly, wobbling around it with decreasing amplitude due to viscous dissipation for $R_o \ll 1$ or wandering in the plane containing the initial position of the pole and the mass anomaly for $R_o \gg 1$. This different behavior is due to the efficiency of the equatorial bulge in stabilizing the rotation of the planet, with $R_o \gg 1$ being indicative of an important equatorial bulge responsible for a slow wandering of the axis of rotation, while the opposite limit favors mega-wobbles around the mass anomaly.

If the time scale for readjustment of the equatorial bulge T_1 and the length of day $T_r = 2\pi\Omega^{-1}$ are well separated and J_2 is large, the rotational number is large, and the planets experience true polar wander; a smaller separation on these time scales and a small dynamic form factor J_2 implies, on the other hand, that TPW is inhibited. For Venus, the smallness of R_o is primarily due to the long rotational period T_r and small dynamic flattening J_2 . Since in this case the centrifugal contribution to the total inertia of Venus is small, we expect that the rotational dynamics of this body is intermediate between that of a rigid

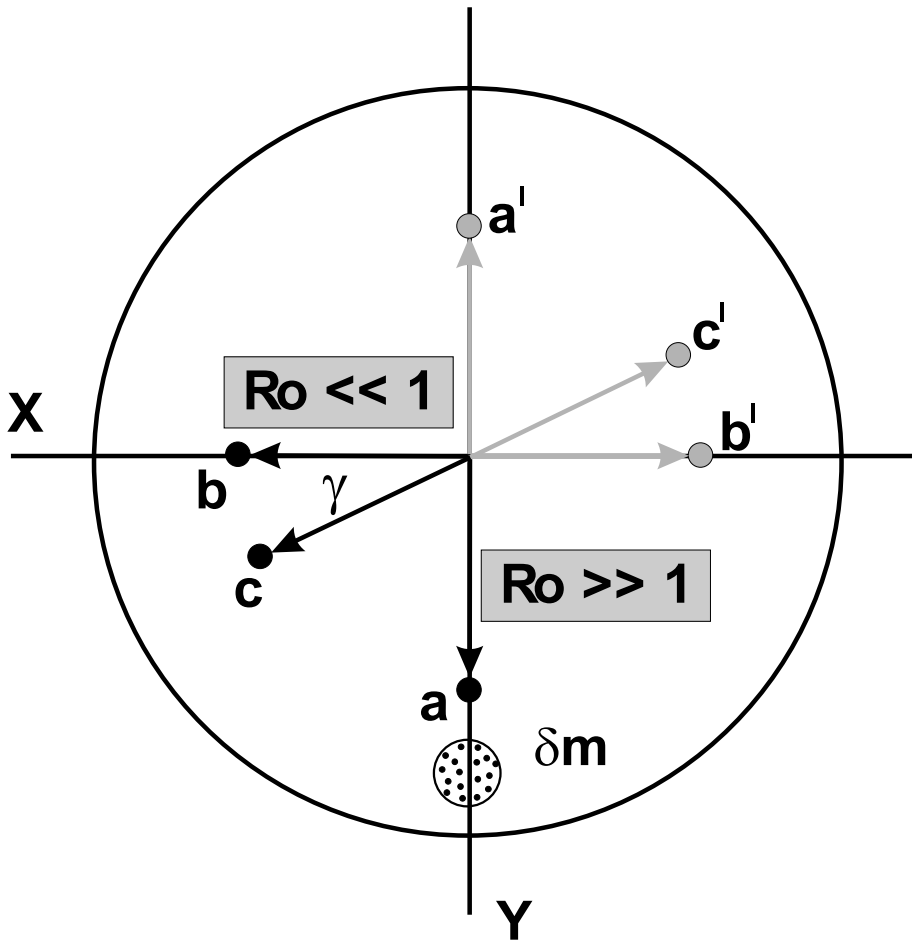


Figure 7.15. Incipient motion of the rotation axis predicted by the Liouville equations for a planet excited by an internal mass heterogeneity δm . For large rotational numbers ($R_o \gg 1$), the pole is moved toward the source along the path Oa (black arrow) if the non-hydrostatic geoid anomaly associated with the dynamic compensation of δm is negative. For positive anomalies, the pole moves in the opposite direction, along Oa' (grey arrow). This behavior is expected for the Earth and Mars, whose R_o clearly exceeds unity (see Figure 7.14). The rotation axis of a planet characterized by $R_o \ll 1$ is initially forced along the direction Ob (or Ob', depending upon the sign of the geoid anomaly). Redrawn from Figure 2 in Spada *et al.* (1996).

body (with no rotational deformation (Lambeck, 1980) or $R_o = 0$ and with the spin axis not necessarily parallel to the main inertia axis) and that of the Earth, whose bulge stabilizes the excursions of the axis of rotation by always trying to keep it parallel to the direction of the principal inertia axis (Ricard *et al.*, 1993). With respect to the time constant T_1 characterizing the readjustment of

the equatorial bulge, the new non-dimensional rotational number R_o defined here accounts simultaneously for the rheological structure of the planet through the T_1 dependence and for the basic rotation through Ω .

The percentage strength r_{2m} of the order m components defining the degree 2 topography for the three planets is shown in Figure 7.16; this strength is defined as

$$r_{2m} = \frac{|t_2^m|^2}{\sum_{k=0}^2 |t_2^k|^2}, \quad (7.36)$$

(Spada *et al.*, 1994) where, for the Earth and Venus, t_2^m are the spherical harmonic degree 2 components of the dynamic topography that are supported by the internal flow driven by mantle sources; for Mars, t_2^m are simply the components of the observed topography, which is not dynamic but induced by surface loads. In rotation dynamics the degree $l = 2$ geoid anomaly component controls the style of long-term polar motion, as discussed in the previous figure, so that, in order to discuss the effects of rotation on the shape of the planets, we must consider the topographic features related to the non-hydrostatic geoid for the three planets. The non-hydrostatic geoid is mostly related to internal mass anomalies for the Earth and Venus and to the observed topography due to surface lithospheric relief for Mars. In order to derive the dynamic topography for the Earth, the harmonic coefficients derived by Cazenave *et al.* (1989) from the observed topography has been used, after removal of the effects due to the cooling of the oceanic lithosphere and of the topography excess in continental areas. The dominant order 2 component is a feature that characterizes the degree 2 of the dynamic topography of the Earth (Cazenave *et al.*, 1989; Ricard and Vigny, 1989) and the observed topography of Mars. As shown in this figure for the Earth, the topography, induced by internal sources, carries up to 88 per cent of the total strength in comparison to 7 per cent for the order zero and 5 per cent of order 1 (Cazenave *et al.*, 1989; Cazenave and Thoraval, 1994; Spada *et al.*, 1994). For Mars, this is mainly due to surface volcanic loads and does not originate from deep-seated internal dynamic processes, such as subduction or density anomalies in the mantle, as indicated by the correlation between the surface topography and the gravity field (e.g., Bills and Ferrari, 1978; Phillips and Lambeck, 1980). The equatorial position of the largest topographic feature of the Martian lithosphere implies a dominant order 2 component in the degree 2 of both topographies, which carries 75 per cent of the strength at this degree (Bills and Ferrari, 1978) and the associated geoid (Sjogren *et al.*, 1975), which shows, in fact, a significant correlation with the Martian topography (Balmino *et al.*, 1982). Due to the topography of the Tharsis region the shape of Mars is thus elongated along an axis lying in the equatorial plane. Unlike Mars, the observed topography of Venus originates essentially from internal processes

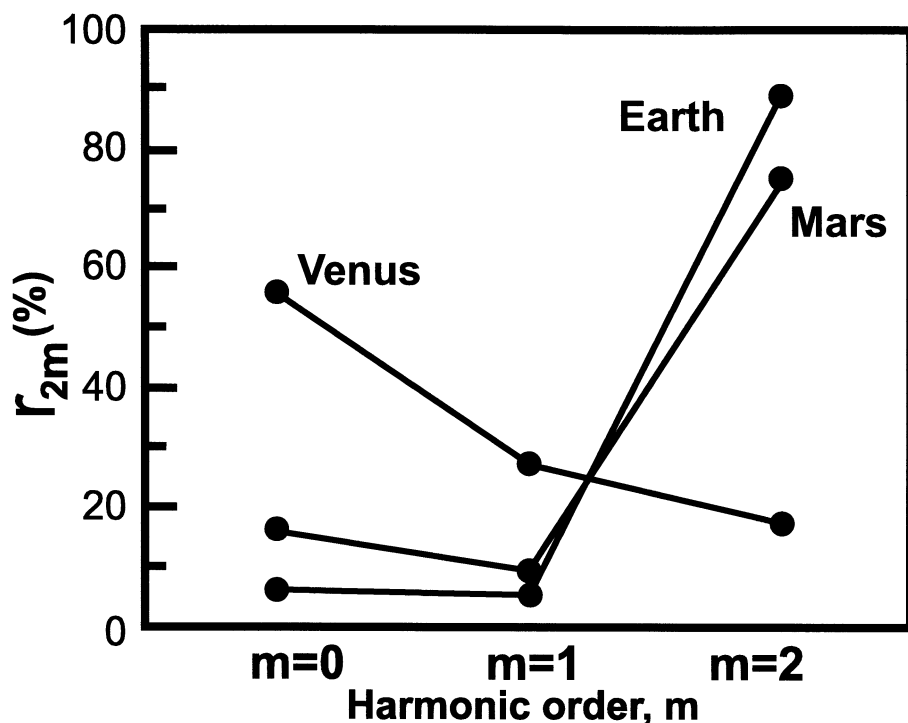


Figure 7.16. Percentage strength r_{2m} of the harmonic degree 2 component for the topographies of the Earth, Mars and Venus. Redrawn from Figure 7 in Spada *et al.* (1996).

and is well correlated with the geoid (Kiefer *et al.*, 1986), so that, to a good approximation, the topography of this planet is the dynamic topography.

This figure suggests a correlation between the dominance of the order 2 component for the Earth and Mars and TPW; Venus, where mega-wobbling is the mechanism for reaching rotation equilibrium rather than polar wander, does not in fact show this feature. As noted in the discussion of the previous figure, Mars can displace the excess topography at the equator by means of polar wander; since the topography is clustered in longitude, the displacement of the excess topography at the equator will induce a large order 2 component. This is similar to what occurs for the Earth, but in a more subtle way, because internal density anomalies rather than surface ones are responsible for the displacement of the rotation pole. On time scales of billions of years, even for the Earth long-term TPW is able to maintain the maxima of the harmonic degree 2 non-hydrostatic geoid at the equator, favoring at the same time the sectorial component of order 2 in the dynamic topography, rather than the zonal component, if the mantle is stratified in viscosity. TPW's internal origin makes the rotational behavior

of the Earth extremely sensitive to the viscosity profile of the mantle; for a viscosity increase in the lower mantle of at least one order of magnitude, in agreement with global geoid models (Ricard *et al.*, 1984; Richards and Hager, 1984) and some postglacial rebound calculations (Spada *et al.*, 1991), positive density anomalies in the mantle like subductions induce at the surface a positive geoid anomaly and a negative dynamic topography, a 'hole' that is carried at the equator together with the positive geoid anomaly. This 'hole' in the dynamic topography at the equator is, of course, responsible for a strong longitudinal dependence in the order 2 component, as observed in the previous figure for the Earth. It is thus a fact that the shape of the Earth agrees well with that of a rotating planet where subduction is the major mechanism driving long-term polar wander and where the lower mantle is stiffer than the upper mantle. This means that polar wander is a mechanism that can modify the figure of the Earth and Mars. The similarity of these two planets in the degree 2 component of the topography appears not to be a simple coincidence but rather related to long-term excursions of the rotation axis. Let us consider as an exception to this rule the case of Venus, which does not have the ability to wander and carries not this signature but rather a large order zero component in the dynamic topography and geoid (Mottlinger *et al.*, 1985; Bills and Kobrick, 1985) that is indicative of an isotropic distribution of mass anomalies within the mantle and zonal symmetry not degraded by TPW. Although the tectonics responsible for the observed morphology on the Earth, Mars and Venus is different (e.g., Head and Solomon, 1981), essentially because subduction is a major mechanism on the Earth, but is inhibited on Mars and Venus by thick (Phillips and Lambeck, 1980) or buoyant (Anderson, 1980, 1981) lithospheres, important analogies can be found in their rotational dynamics.

This walk on terrestrial planets has shown that Mars and the Earth belong to the category that allows long-term polar wander by means of a slow walk of the pole of rotation associated with the readjustment of the equatorial bulge; Venus belongs to the second category, where a perturbation in the moment of inertia induces a mega-wobble due to the smallness of the equatorial bulge, the axis of rotation and maximum inertia not necessarily being aligned on long time-scales. Although the Earth and Venus belong to the two different categories, their driving mechanism comes from internal sources, unlike Mars, where lithospheric phenomena are the main sources of polar wander. The role played by the viscous properties of the mantle of the three planets, whose viscosities are within one or two orders of magnitude from one another, is crucial to our analysis. The similarity in the viscosity structure is the common characteristic enabling us to elucidate the peculiar features in their rotation, the other ingredient being the length of the sidereal day that determines the size of the equatorial bulge. The equatorial location of the volcanic complex on Mars is thus likely to be a consequence of polar wander, as indicated by our quantitative

analysis in this section. This finding allows us to elucidate the influence of TPW on the degree 2 pattern of the topography associated with the volcanic complex (Willeman and Turcotte, 1982) and to draw a comparison with the other two planets considered in our analysis. The rotation parameters and the rheology of the mantle can be conveniently expressed by means of a non-dimensional rotational number that tells us when a planet experiences TPW. There is thus a close connection between different aspects of the dynamics of terrestrial planets, which involve apparently separate realms of planetary physics such as rheology of the interiors, rotation, gravity, surface tectonics, and internal processes.

Chapter 8

POST-SEISMIC DEFORMATION

1. GLOBAL POST-SEISMIC DEFORMATION

The occurrence of faulting in the lithosphere is responsible for an instantaneous deformation of the Earth's surface, which is called co-seismic deformation. In response to the induced elastic stress in the mantle, creep occurs in this deep portion of the planet, or in the ductile portion of the crust, via viscoelastic stress relaxation. Immediately after the occurrence of the earthquake, the mechanism of stress release due to viscous flow in the ductile part of the Earth's crust starts to operate, leading to post-seismic deformation. The delayed deformation of the lithosphere caused by stress relaxation in the mantle or in the low viscosity layers of the crust is called post-seismic deformation. In this section we deal with the effects of a realistic PREM-based (Dziewonski and Anderson, 1981) lithosphere and mantle stratification on post-seismic deformation after a series of preliminary analyses based on simplified models carrying at most four layers, namely the lithosphere, the upper and lower mantle and the core. Relaxation is here considered to occur in the mantle, while in the following section the effects of stress relaxation in the ductile portion of the crust will be considered.

Our analysis is key to the understanding of plate-mantle interaction, especially for the correct interpretation of geodetic VLBI and GPS data of plate deformation when earthquakes occur within the plate or at its boundaries. Improvements in satellite differential radar interferometry (Massonnet *et al.*, 1993) or in precise GPS monitoring of crustal motions necessitate realistic and precise models of co-seismic and post-seismic effects, which must include sphericity, self-gravitation and stratification of the lithosphere and mantle, as first outlined in the present analysis based on a completely analytical approach. The explicit expression of the body force equivalents $\mathbf{f}$ and $\mathbf{f}'$ that must be inserted in the

general normal mode solution, equation (1.120), in order to obtain the solution vector $\mathbf{y}(r)$, is given in Appendix C. In order to study post-seismic deformation we have to go up to harmonic degrees of the order of 10^3 , one order of magnitude higher than in post-glacial rebound, as the peaks of the total strength of the modes as a function of zonal degree can be situated at zonal degrees of several thousands for shallow earthquakes in the upper crust, as considered in the following section.

With respect to post-glacial rebound, the toroidal solutions of sections 1.3.2 and 1.5.2 must also be considered. The procedure for solving post-seismic problems in viscoelasticity, as shown in section 1.8.1, was first described by Sabadini *et al.* (1984a).

Figure 8.1, dealing with the relaxation times as a function of harmonic degree, shows that the modes form nice continuous-like patterns with increasing harmonic degree. The model used in this calculation is a 10-layer model, averaged from PREM, in which the uppermost mantle is stratified into four viscoelastic layers, with varying density and rigidity but uniform viscosity of 10^{21} Pa s; the transition zone and lower mantle are homogeneous. The model is presented in Table 8.1.

The fastest transient viscoelastic modes superimpose at the bottom of the scale. For all the models shown here carrying a different number of layers, all the theoretically predicted modes have been successfully detected, which guarantees that the basis upon which our fields are developed is complete. It should be noted too that not all of the theoretically predicted 17 modes are important. For example, the top four modes for each harmonic degree in Figure 8.1, which are the internal-mantle buoyancy modes, have extremely long relaxation times (some even exceed the age of the Earth) and extremely small strengths. Still, their detection is useful in checking that all modes which do have non-negligible strengths have been detected.

In order to elucidate the influence of lithospheric and mantle stratification on post-seismic deformation, we show in the following figures the displacement pattern for a vertical, point-like, dip-slip source embedded at 100-km depth at the base of the lithosphere; the seismic moment of the source is fixed at 10^{22} Nm, characteristic of a large earthquake. Radial displacement fields are sampled at an azimuth of 90° with respect to the strike of the fault, in the subsiding portion of the Earth's surface; tangential displacements are sampled at 45° . At 180° from the direction that we have considered, we would have obtained the same displacement patterns, but with the reversed sign. Figure 8.2 shows, for a fixed distance of 200 km from the epicenter of the fault and varying time after the occurrence of the earthquake, the radial and tangential displacement components, (a) and (b) respectively. Remarkable differences can be observed among the 4- and 10-layer volume-averaged models and model P1 as used in previous analyses by Piersanti *et al.* (1995) and Sabadini *et al.* (1995),

Table 8.1. Parameters for the 10-layer volume-averaged earth model. r is the distance with respect to the center of the Earth, ρ the density of the layer, and μ the rigidity. The viscosity ν is given for the convex profile models (Figure 2.3).

Layer	r (km)	ρ (kg/m ³)	μ (N/m ²)	ν (Pa s)	
1	6371 – 6356	2283	2.66×10^{10}		elastic lithosphere
2	6356 – 6331	3194	5.91×10^{10}		
3	6331 – 6251	3372	6.77×10^{10}		
4	6251 – 6221	3372	6.69×10^{10}	1.5×10^{20}	uppermost mantle
5	6221 – 6151	3372	6.61×10^{10}	2.5×10^{20}	
6	6151 – 6061	3462	7.56×10^{10}	4.5×10^{20}	
7	6061 – 5971	3515	7.89×10^{10}	7.0×10^{20}	
8	5971 – 5701	3857	1.06×10^{11}	3.0×10^{21}	transition zone
9	5701 – 3480	4878	2.19×10^{11}	2.4×10^{22}	lower mantle
10	3480 – 0	10932			inviscid fluid core

Table 8.2. Parameters for the 4-layer volume-averaged earth model. r is the distance with respect to the center of the Earth; ρ the density of the layer, and μ the rigidity. The viscosity ν is given for the convex profile models.

layer	r (km)	ρ (kg/m ³)	μ (N/m ²)	ν (Pa s)	
1	6371 – 6250	3234	5.99×10^{10}		elastic lithosphere
2	6250 – 5701	3631	8.60×10^{10}	1.6×10^{21}	upper mantle
3	5701 – 3480	4878	2.17×10^{11}	2.4×10^{22}	lower mantle
4	3480 – 0	10932			inviscid fluid core

in which the rigidity and density parameters were taken from fixed-boundary contrast models. The parameters for the 4-layer volume-averaged model are given in Table 8.2; the parameters for model P1 can be found in Table 1 of Piersanti *et al.* (1995).

For radial displacement, panel (a), the volume-averaged models predict an increase in the post-seismic deformation; model P1 (short-dashed curve) predicts the opposite behavior, a reduction in the post-seismic signal. Although similar to the 10-layer model in the amplification of the post-seismic deformation,

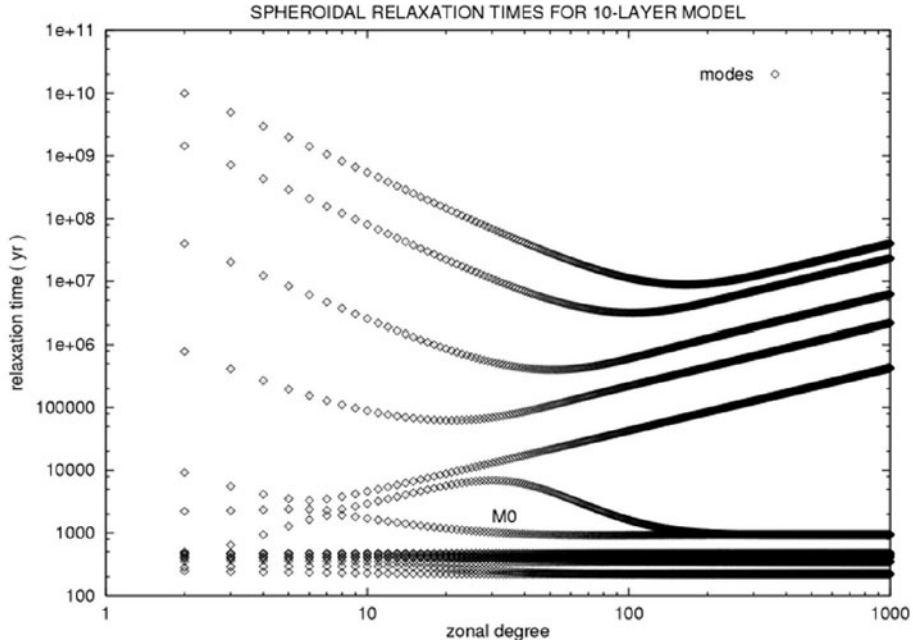


Figure 8.1. Relaxation times, in years, as a function of the harmonic zonal degree for the 10-layer model averaged from PREM and described in Table 8.1; the elastic lithosphere has three layers and the uppermost mantle is divided into four viscoelastic layers with a viscosity of 10^{21} Pa s, which is also the value for the viscosity of the homogeneous transition zone and lower mantle. The fundamental mantle modes are indicated by *M0* (Figure 1 in Sabadini and Vermeersen, 1997).

the 4-layer model (solid curve) predicts, both in the elastic and long time-scale limit, a radial displacement which is higher by a factor two in comparison to the 10-layer models (dashed curve). While the behavior of the old P1 model is due to the absence of elastic stratification between the lithosphere and the mantle, responsible for an overestimated lithospheric rigidity, deviations in the 4-layer model with respect to the 10-layer one are due to the absence of stratification in the elastic lithosphere in the 4-layer model. This finding clearly indicates the necessity of stratifying the lithosphere in global post-seismic deformation models.

This result is not surprising, since the final state, after relaxation has taken place in the mantle, depends solely on the elastic properties of the lithosphere. We have verified that a further refinement of the lithospheric layering does not modify the results with respect to the 10-layer model. The largest rigidity and density contrasts within the lithosphere are located at the base of the upper and lower crust, with the remaining portion of the lithosphere essentially ho-

mogeneous, which guarantees that a three-layered lithosphere is sufficient to provide realistic estimates of the effects of earthquakes on global co-seismic and post-seismic deformation.

In panel (b) of Figure 8.2, a similar analysis is carried out for the tangential displacement, measured in the outward direction with respect to the dislocation, at 45° from the strike of the fault. On analogy with what is obtained in panel (a), the tangential displacement of the P1 model is also subject to a substantial reduction during post-seismic deformation. This is in distinct contrast with the results of the volume-averaged models, which show a small reduction in the signal with respect to the co-seismic deformation. Comparison between the 4- and 10-layer models reinforces the conclusion drawn from the analysis in panel (a) of the importance of stratifying the outer elastic layer of the Earth for realistic estimates of post-seismic displacements.

Figure 8.3 shows the displacement as a function of the distance from the epicenter for the same models in Figure 8.2, but for fixed time intervals after the earthquake; the 4- and 10-layer viscosity models are averaged from the convex viscosity profile of Figure 2.3. The viscosity values can be found in Tables 8.1 and 8.2. The panels depict three snapshots of the deformation: panel (a) for the elastic limit, and panel (b) for an intermediate time at which the post-seismic deformation is still evolving and for the long time-scale limit. The most striking result in this figure is that the 10-layer model behaves in a completely different fashion from the simplified P1 and 4-layer models, in all three time snapshots. In the elastic limit, the P1 and 4-layer model overestimate the radial displacement by more than a factor of two. During the transient at $t = 10^3$ yr, the P1 model predicts a substantial reduction rather than a smooth amplification of the post-seismic signal, as predicted by the volume-averaged 4- and 10-layer models. In the far field, we notice an upwarping of the lithosphere for the 10-layer model, which is not predicted by the 4-layered model, due to the overestimated rigidity of the outer elastic layer. In the long time-scale configuration, the 10-layer model does not show a visible increase in the deformation in the near field, while in the far field this model has rebounded with an annihilation of the upwarping that was noted during the transient regime. This result indicates, as expected, that mantle relaxation has a strong control of post-seismic deformation in the far field from the epicenter of large earthquakes, and thus ultimately controls plate-mantle interaction. In this final state, the old P1 model predicts a pattern which does not show any resemblance to the more realistically stratified models, providing a smooth deformation of a few centimeters, in comparison with the 50 cm at 50 km from the epicenter as predicted by the 10-layer model. Except for the post-seismic deformation in the far field noted above, the drastically different behavior of the three models is due to whether the lithosphere has been stratified or not.

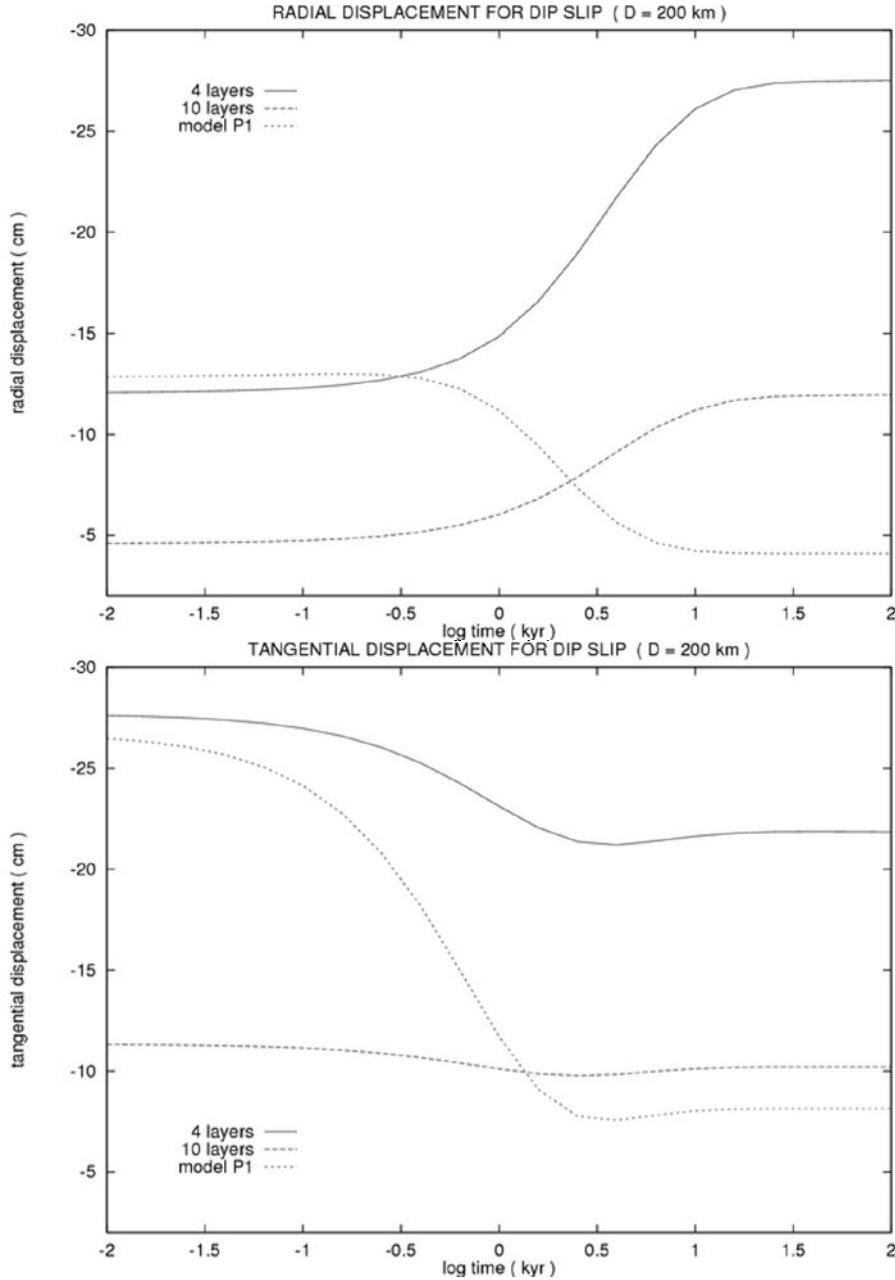


Figure 8.2. Radial displacement, top panel, and tangential displacement, bottom panel, in centimeters as a function of time in years after the occurrence of the faulting, at $t = 0$ yr; the source is a vertical, point-like dip-slip, with a seismic moment of 10^{22} Nm. Negative values denote subsidence; the distance from the epicenter of the fault is $D=200$ km. The parameters for the 4- and 10-layer volume-averaged models are given in Tables 8.2 and 8.1; model P1 is given in Table 1 of Piersanti *et al.* (1995). Viscosity is fixed at 10^{21} Pa s in each viscoelastic mantle layer of all models (Figure 2 in Sabadini and Vermeersen, 1997).

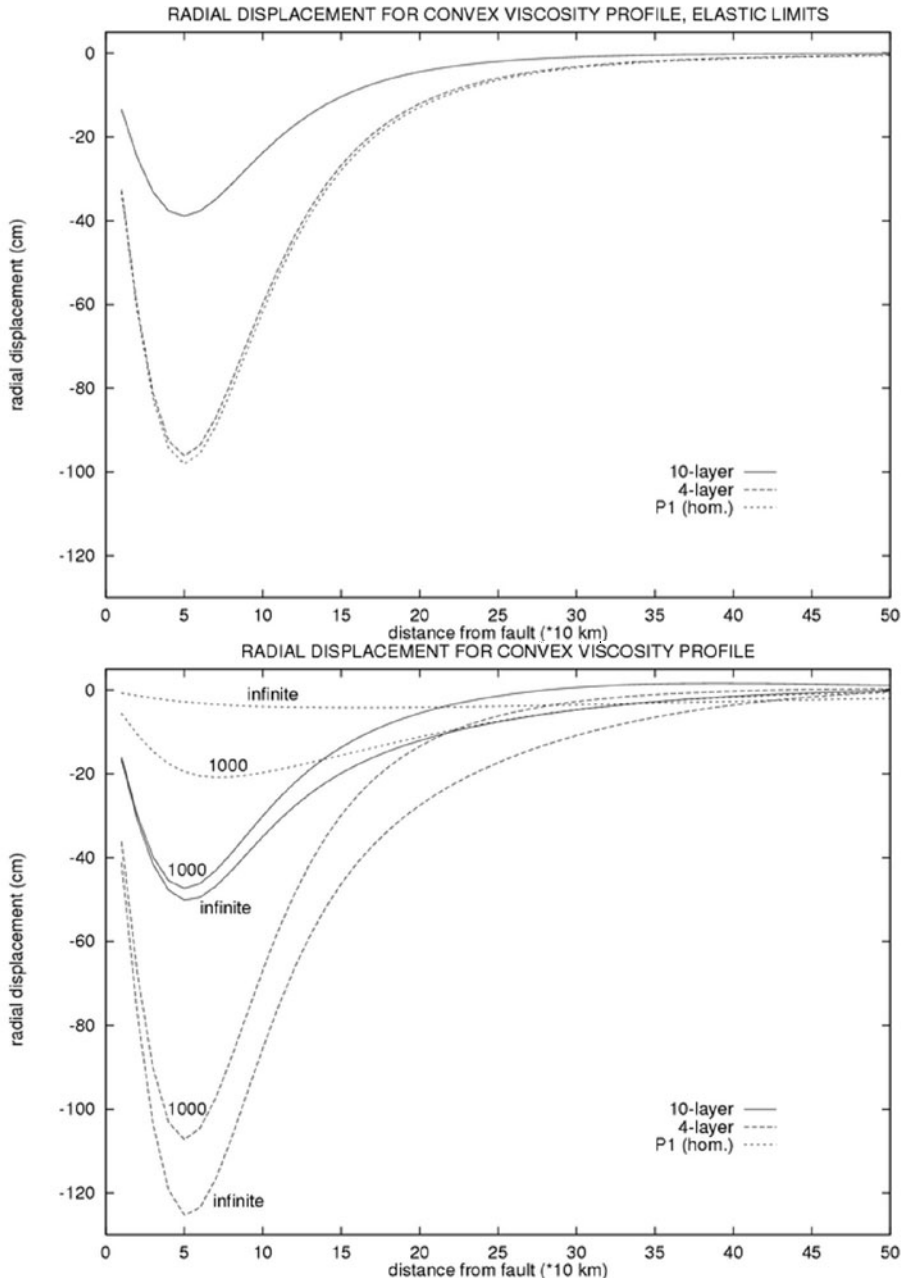


Figure 8.3. Radial displacement as a function of the distance from the fault in the elastic limit, top panel, and at $t = 1,000$ yr after the earthquake together with the long time-scale $t = \infty$ limit, bottom panel. The curves depict the cases for the 10-layer model, solid curve, 4-layer model, dashed curve, and model P1, short-dashed curve. The mantle viscosity of the 4- and 10-layer models is given in Tables 8.2 and 8.1. Model P1 has a uniform mantle viscosity of 10^{21} Pa s. In all cases the lithosphere is elastic (Figure 3 in Sabadini and Vermeersen, 1997).

Figure 8.4 shows the effects of mantle viscosity stratification based on the comparison between two 10-layer models, with the same lithospheric stratification, for a uniform mantle viscosity of 10^{21} Pa s and convex viscosity profile. The elastic and long time-scale limits of the homogeneous and convex viscosity mantle are the same (short-dashed and dotted curves, respectively): being identical, the elastic profiles of the mantle and lithosphere control the initial and final configurations. While in proximity to the epicenter, at distances in the order of 10 – 100 km, the two models behave in a similar fashion (except for a 10 per cent smaller signal at 50 km for the homogeneous model caused by a higher viscosity in the upper mantle), in the far field the homogeneous and convex mantle viscosity models produce a quite different response at intermediate time-scales of 1 – 1000 yr. As already noted in the discussion of panel (b) in Figure 8.3, displacement in the far field occurs in the opposite direction with respect to that experienced in the near field. The convex viscosity profile model predicts, in fact, positive vertical displacements of about 3 cm, while the homogeneous viscosity mantle model does not show any visible evidence of relaxation, still providing the values of co-seismic deformation. This result clearly indicates the importance of mantle rheological stratification in controlling the deformation of the plate in the far field during post-seismic deformation. The upward deflection of the lithosphere in the far field caused by the convex viscosity mantle is due to the upper mantle, which is softer than in the homogeneous model.

The elastic stratification of the lithosphere has a major influence on post-seismic deformation; at least three layers in the lithosphere, averaged from PREM, are necessary to obtain a correct estimate of the deformation following large earthquakes, both in the near field and in the far field. We have verified that a further increase in the eleven lithospheric layers of PREM produces a minor refinement in post-seismic deformation estimates in comparison with the major improvement that is gained from one to three lithospheric layers. The modification in the deformation pattern during the evolution from the initial elastic state to the long time-scale one is controlled by mantle rheology: viscoelastic stratification of the mantle has a major influence on the rebound of the lithosphere in the far field following the earthquake. In comparison with the required stratification of the lithosphere, the same fineness of layering is not needed in the mantle, in agreement with the result that mantle rheological stratification controls the global pattern of post-seismic deformation in the far field, not being sensitive to the short wavelength features of the mantle structure.

2. POST-SEISMIC DEFORMATION FOR SHALLOW EARTHQUAKES

In the previous section large and deep earthquakes were considered in order to elucidate the global properties of post-seismic deformation by focusing in

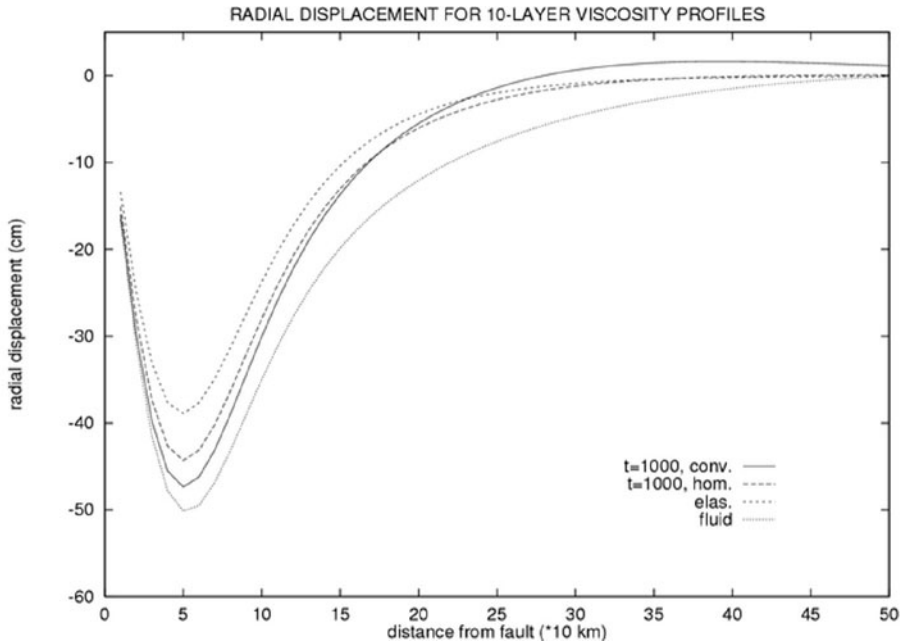


Figure 8.4. Comparison between a 10-layer model with a uniform viscosity of 10^{21} Pa s, dashed curve, and convex viscosity, solid curve, at $t = 1,000$ yr after the earthquake. The short-dashed and dotted curves denote the elastic and long time-scale limits of the two models (Figure 4 in Sabadini and Vermeersen, 1997).

particular on stress relaxation in the mantle. In this section the normal mode theory is applied to carry out the effects of stress relaxation in the lower crust. By dealing with shallow sources and rheologically stratified crusts, the normal mode expansion must be performed to high harmonic degrees, up to $l = 6,000$, to resolve post-seismic displacements for earthquakes whose faults cut a thin upper elastic crust overlying a viscoelastic lower crust. It is possible to show that the high resolution in co-seismic displacements necessitates summation to over 4×10^4 spherical harmonics, while for estimating rates of deformation, due to stress relaxation in the lower crust, summation over spherical harmonics of a few times 10^3 is sufficient. Improvements in the normal mode theory for dealing with harmonic degrees in the order of 10^4 is given in section 1.9. The models shown in this chapter are totally new with respect to those which stemmed from Piersanti *et al.* (1995) in terms of both the large number of layers dealt with in the present book, first made possible by the grid-spacing procedures described in section 2.1, representing the real heart of this new family of post-seismic deformation models, and by the high spatial resolution based on the rescaling procedures of the viscoelastic solution described in section 1.9.2.

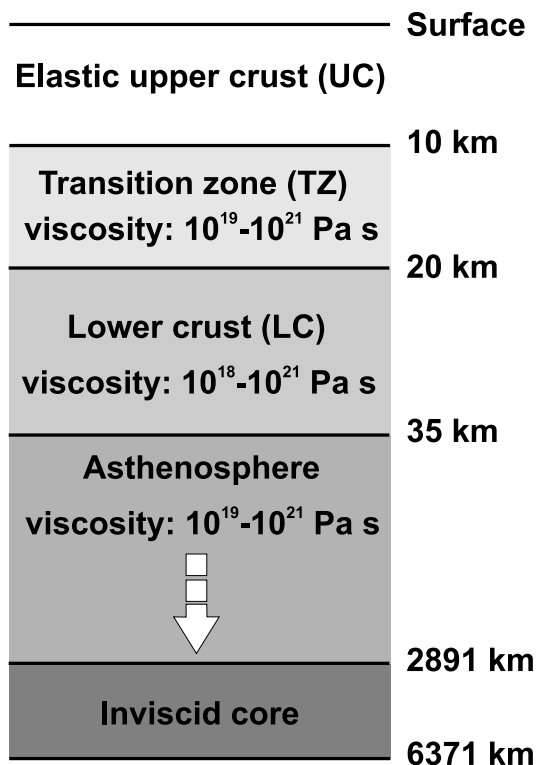


Figure 8.5. Earth model and viscosity parameters used for the Umbria-Marche earthquake, redrawn from Figure 1 in Riva *et al.* (2000).

2.1 THE UMBRIA-MARCHE (1997) EARTHQUAKE

So as to show the effects of stress relaxation in the lower crust, the Umbria-Marche 1997 moderate-size earthquake, characteristic of a slowly deforming plate boundary in the central Mediterranean, is modeled in this section 8.2.1, reworked from Riva *et al.* (2000), by means of the normal mode theory. Central Italy is undergoing continental extension and experienced the moderate Umbria-Marche 1997 normal faulting earthquake sequence. Deep seismic reflection studies (CROP03) and the 1997 earthquake sequence clearly show a seismogenic layer decoupled from the lower crust by a sizeable transition zone. In accord with these observations, the modeling is based on a crust divided into three layers: an elastic upper crust, a transition zone and a low-viscosity lower crust, as shown in Figure 8.5.

The fault is embedded in the upper crust. The calculation of post-seismic deformation requires the assumption of a layered viscoelastic earth model. Nearly all viscoelastic relaxation studies (Ma and Kuszniir, 1995; Pollitz, 1996) assume a three-layer model schematizing the brittle upper crust (UC), the lower crust

(LC) and the mantle. Seismological and geological observations from regions of active faulting suggest that the seismogenic UC is likely to be separated from the LC by a transition zone (TZ) where both brittle and ductile processes can occur (Sibson, 1982; Meissner and Strehlau, 1982; Chen and Molnar, 1983, Sibson, 1989; Scholz, 1990). These observations are more frequent in regions of active continental extension (Jackson and White, 1989). The recent CROP03 seismic reflection profile (Pialli *et al.*, 1998), designed for the study of the deep crust in the northern Apennines, imaged the top of the fully ductile LC using regional high-amplitude reflectors. Below the Umbria-Marche extensional belt, the top of the LC is about 20 km deep, with a Moho around 35 km (Coli, 1998). Cattaneo *et al.* (2000) show an abrupt cutoff of the 1997 aftershock sequence at about 9 km of depth. Thus the seismic activity is largely restricted to the uppermost 10 km of the deforming continental crust and leaves room for a 10 km-thick TZ above 15 km of LC as imaged by the CROP03 section (Figure 8.5). The TZ acts in the normal mode modeling as a layer which decouples the elastic UC from the low-viscosity LC. Accordingly, the modeling is based on five layers (Figure 8.5). The relevant stratification is consistent with the range of regional velocity models estimated via the analysis of surface-wave dispersion (Calcagnile and Panza, 1981) and regional gravity and heat flow anomalies (Della Vedova *et al.*, 1991; Marson *et al.*, 1995) and other relevant geological and geophysical data (Du *et al.*, 1998). The Umbria-Marche earthquake sequence started on September 26, 1997, and took place in a complex deforming zone along a normal fault system in the central Apennines. This shallow sequence was followed by a mantle event on March 26, 1997 (Mw 5.2) located at a depth of 46 km (Saraó *et al.*, 1998; Aoudia *et al.*, 2000). Only the strongest earthquake of the Umbria-Marche sequence that took place on the 26 September 1997 at 9:40 (Ms = 6.0) is considered in the modeling. The seismic moment is based on the solution in Saraó *et al.* (1998) retrieved by broad-band waveform inversion, in good agreement with the CMT solution (Ekstrom *et al.*, 1998). The application below makes it possible to elucidate the characteristics of post-seismic deformation in the studied region and to provide a first-order estimate of the expected vertical velocities.

To illustrate the sensitivity of the results on the depth of the source, the finiteness of the source in the vertical direction is neglected, and besides the source depth of 7 km (Saraó *et al.*, 1998, Aoudia *et al.*, 2003), a case in which the source is embedded at a depth of 3 km is also considered to simulate the effects of a fault nearly reaching the surface. We distribute the seismic moment along a line of dislocations whose length along strike is 10 km; the source has thus an infinitesimal extension (a point) in the down dip direction. The dip of the fault is 37°, in agreement with the Saraó *et al.* (1998) solution. In order to resolve the deformation produced by this fault, summation of 4000 normal modes is performed, which ensures convergence of the solution. The vertical

deformation is considered in the following simulations, being most relevant to normal faults; the results are evaluated at the surface along a line perpendicular to the strike of the fault, crossing its center. In the whole set of panels, the fault is dipping to the left, and the distance is measured from the intersection of the upward prolongation of the fault with the Earth's surface. In Figure 8.6, the depth is fixed at 7 km.

Figure 8.6a portrays the co-seismic and post-seismic vertical deformation, measured at the surface. The co-seismic component (solid line) shows the deformation pattern characteristic of normal faulting, with an uplifted, localized footwall and a broad subsidence in the hanging wall. The post-seismic component (dashed line), which includes the co-seismic one plus the extra deformation resulting from the relaxation in the viscoelastic medium, is responsible for a broadening of the subsidence, which also affects the footwall. Post-seismic displacements are recorded after complete relaxation occurred in viscoelastic layers. The largest co-seismic displacement is -8 cm 10 km from the fault. Due to the relaxation, the maximum displacement increases to -12 cm, and a broad area, which remained at rest during the co-seismic deformation, is now subject to subsidence at distances of tens of kilometers. This effect is due to the decoupling of the uppermost part of the crust with respect to the lower layers. The occurrence of normal faulting and the extension at the bottom of the UC, below the neutral plane, is responsible for the down flexure of this layer, visible in the broad subsidence in Figure 8.6a.

Figure 8.6b provides another perspective to this physics, with the vertical velocities expected at different times after the earthquake. The wide region of negative velocity marks the broad down-flexure of the UC. The two regions of low velocity represent the peripheral response of the layer to the subsidence. Another interesting characteristic in Figure 8.6b is the short wavelength feature at the center of the subsiding area: we can recognize the relative maximum corresponding to the footwall and the largest subsidence of the hanging wall, within a generalized subsidence. The highest velocity is -0.85 mm/yr in the hanging wall, immediately after the earthquake, at $t = 0$. Due to the relatively high viscosity of the TZ (10^{19} Pa s), this velocity shows a minor reduction after 10 yr, and after 50 yr there is still a visible signal of -0.25 mm/yr. If, for the TZ, a viscosity of 10^{18} Pa s is also considered, hence coupled with the LC, the highest velocity increases by one order of magnitude to -8.5 mm/yr. Since the viscosity of the TZ and LC is largely uncertain, probably ranging between 10^{18} Pa s and 10^{19} Pa s, it is likely that the largest subsidence rates in the highly deforming area of the hanging wall vary in the range that we have estimated. Comparison between the ongoing GPS campaigns and model predictions is crucial in estimating the effective viscosity of the TZ and LC.

Figure 8.6c deals with stress relaxation limited to the mantle, with sampling times increased with respect to the previous cases. The deformation pattern

is broadened since stress relaxation involves a larger portion of the Earth and the global subsidence disappears because the whole crust is now coupled. The amplitude of the vertical velocity in the earthquake area is substantially reduced since the whole planet is now involved in the deformation, as indicated by the slow decay at increasing distance from the epicentral area and longer relaxation times with respect to Figure 8.6b. Figures 8.6b and 8.6c show that for shallow normal faulting the TZ and LC play a major role in comparison to stress relaxation in the mantle, which is actually negligible.

In Figure 8.7 the source is embedded at a depth of 3 km, in the top half of the UC. In comparison with Figure 8.6, the pattern of both co-seismic and post-seismic displacement in Figure 8.7 becomes sharper, and the largest co-seismic subsidence in the hanging wall is increased by a factor of two with respect to the case of 7 km of depth (Figure 8.6a). The uplift in the footwall, with respect to Figure 8.6a, is subject to a larger increase, by a factor of three. The difference between the highest uplift in the footwall and subsidence in the hanging wall is thus doubled with respect to Figure 8.6a. This increase in the amplitude of the displacement and sharpening of its pattern is attributable to the decrease in the wavelength caused by the shallower source depth in comparison with Figure 8.6. The most interesting difference with respect to Figure 8.6 is the upward migration of the post-seismic displacement pattern with respect to the co-seismic one. An extensional source located above the neutral plane of the UC causes a bending moment opposite to the one induced by the deeper source in Figure 8.6, and the Earth's surface is thus subject to a general uplift, rather than subsidence. From Figures 8.6a and 8.7a, depending on the location of the source beneath or above the neutral plane, the post-seismic curves lie completely above or below the co-seismic one. These findings differ from those reported by Ma and Kusznir (1995) where, due to the finiteness of the source in the vertical direction, the fault cuts the whole UC through its neutral plane, which causes the crossing of the curves depicting the post-seismic and co-seismic vertical displacement.

When attention is drawn to the velocity pattern in Figure 8.7b, the global upwarping observed in the post-seismic displacement also affects the vertical velocity, in agreement with the previous observations on the flexural properties of the UC. Uplift velocities of 1.2 mm/yr are in fact obtained in the hanging wall region, close to the footwall. This tendency of a general upwarping is visible also in Figure 8.7c, where relaxation involves only the mantle. In comparison with Figure 8.6c, we note an increase in the uplift velocity. For Figure 8.7c the same observations about Figure 8.6c can be made, except that now the uplift of the footwall is enhanced with respect to the subsidence in the hanging wall.

The detection of post-seismic deformation in the Umbria-Marche area is an extremely challenging task due to the smallness of the expected signal, of the order of a few millimeters per year, for both vertical and horizontal components.

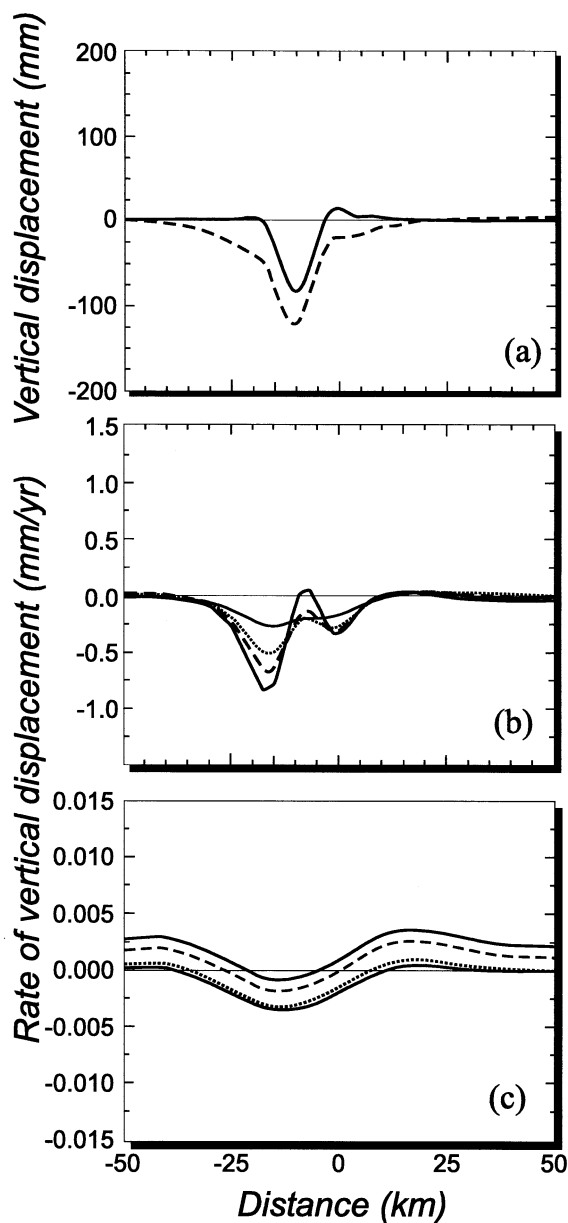


Figure 8.6. Co-seismic and post-seismic vertical surface displacements and rates calculated for a 7-km deep normal faulting line-source, redrawn from Figure 2 in Riva *et al.* (2000). The profiles are across the center of the fault ($x = 0$ km) and perpendicular to its strike. The fault is dipping to the left and is buried in the upper layer of Figure 8.5. (a) Co-seismic (solid line) and post-seismic (dashed line) displacements. (b) Post-seismic rates due to relaxation in the TZ expected at different times after the earthquake, with $t = 0, 10, 20, 50$ yr, solid, dashed, dash-dotted and dotted, respectively. (c) Post-seismic rates due to relaxation in the mantle expected at $t = 0, 100, 500, 1000$ yr, solid, dashed, dash-dotted and dotted, respectively, after the earthquake.

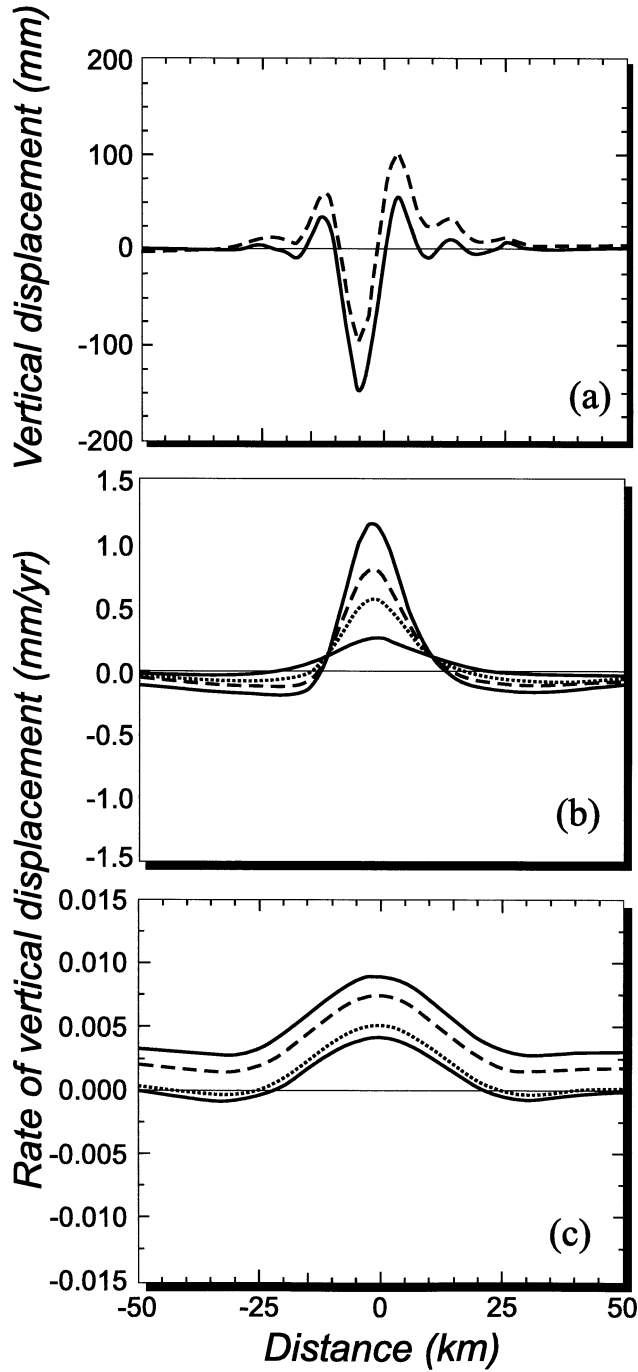


Figure 8.7. Co-seismic and post-seismic vertical surface displacements and rates calculated for a 3-km deep normal faulting line-source, redrawn from Figure 3 in Riva *et al.* (2000). Terminology as in Figure 8.6.

In order to reveal post-seismic effects in the area, a series of GPS campaigns has been undertaken in order to detect in particular the horizontal deformation components. The results of this surveying, along a baseline perpendicular to the major fault activated during the 1997 Umbria-Marche seismic sequence, is shown in Figure 8.8. In order to provide results independent from the reference system, the displacements are evaluated with respect to the town of Spello, indicated by A_0 . These results represent a first evidence of horizontal post-seismic deformation for shallow earthquakes in the Mediterranean region, as shown in Aoudia *et al.* (2003). Figure 8.8, redrawn from Aoudia *et al.* (2003), provides the comparison between the GPS baseline variations and the modeled ones as a function of the baseline length in kilometers from the reference site of Spello (A_0). All baseline variations are positive, except $A_0 - A_1$, which indicates a general trend of post-seismic extension across the fault. Bernese and GIPSY estimates are shown as one sigma error bars and the model results correspond to a transition zone viscosity of 10^{18} Pa s and a lower crust viscosity of 10^{17} Pa s using different fault models, indicated by FM1, FM2 and FM3, as indicated in the caption. FM1 (dotted) is not adequate to explain the observations at different sites and does not reproduce the general pattern of baseline variations shown by the two GPS analyses. FM2 (dashed) reproduces only part of the observations and the general trend, while FM3 (solid) is the best performing one, being able to reproduce almost all the observations and the general baseline variation pattern. These findings demonstrate the important role of GPS survey in seismogenic zones in revealing the physics of post-seismic deformation and providing a fundamental tool in modern seismic hazard studies.

The results of this section provide a first estimate of the expected rates of vertical post-seismic deformation in the Umbria-Marche epicentral area. Fine rheological layering in the crust plays a major role in the interplay between relaxation in the TZ and LC with respect to relaxation in the mantle as shown in section 8.1 when shallow and moderate normal faulting earthquakes are considered in the normal mode theory. Unlike large and generally deep earthquakes (Thatcher and Rundle, 1979; Melosh, 1983; Pollitz and Sacks, 1997; Suito and Hirahara, 1999), shallow and moderate normal-faulting earthquakes are free of the influence of lower lithospheric viscous coupling, while the TZ and low-viscosity LC impose a pattern and a scale on post-seismic deformation. For normal faulting, these applications show the relevance of the depth of the source with respect to the thickness of the elastic layer in controlling post-seismic upwarping or downwarping, as emphasized by Thatcher and Rundle (1979) and Melosh (1983) for thrust faults. Since the elastic layer is the upper crust rather than the whole lithosphere, these findings show that the results obtained by Thatcher and Rundle (1979) and Melosh (1983) are scale-invariant and applicable to dip-slip sources embedded in an elastic layer of any thickness overlying a viscoelastic one. Furthermore, the pattern of widening and narrowing of the

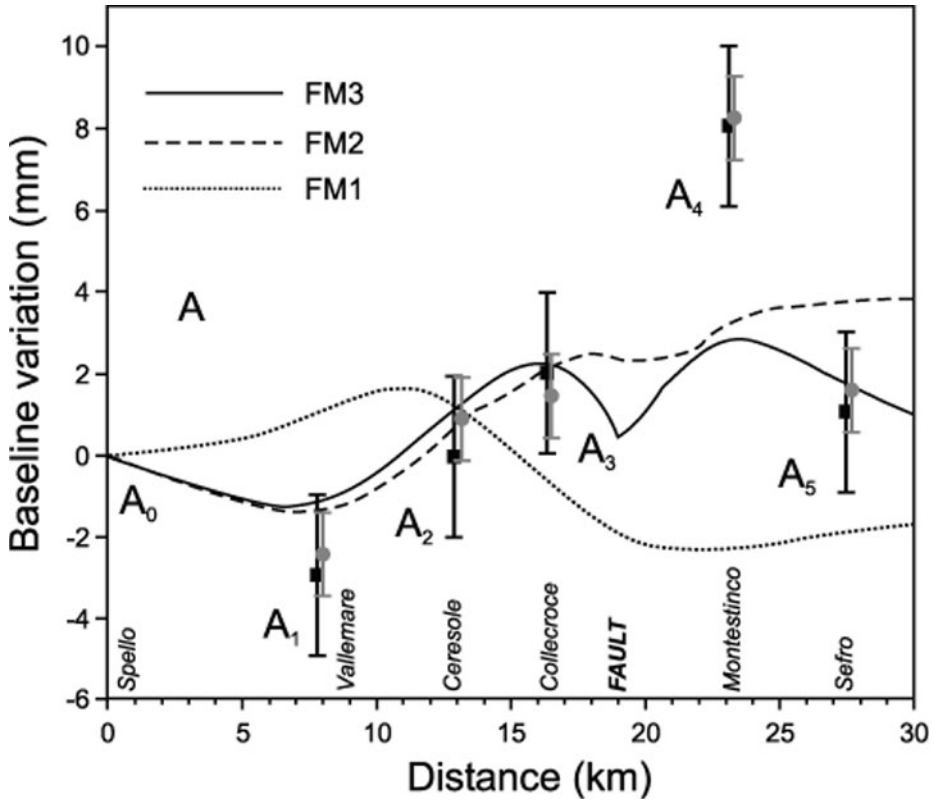


Figure 8.8. 2002-1999 baseline length variations along the GPS section with respect to the A_0 site using GIPSY and Bernese solutions, grey and black vertical bars, respectively, vs. viscoelastic model predictions for the fault model FM1 (Salvi *et al.*, 2000), FM2 (Zollo *et al.*, 1999) and FM3 (Basili and Meghraoui, 2001) fault models, using a viscosity of 10^{18} Pa s and 10^{17} Pa s for the transition zone and lower crust, respectively.

deformation due to stress relaxation at different depths (Figures 8.6 and 8.7) is in agreement with the results obtained by King *et al.* (1988) using a simpler model. These findings show that an appropriate test area for applying the viscoelastic normal mode theory at high harmonic degrees and for studying the effects of the TZ and LC is Italy's central Apennines, where a clear cutoff in the depth distribution of earthquakes indicates that the seismogenic layer is limited to the first ten kilometers of the crust and a well-developed TZ lies below this depth, above a low-viscosity LC.

Table 8.3. Viscoelastic model parameters for the 1980 Irpinia earthquake (UC=upper crust; LC=lower crust; LCB=lower crust bottom; M=mantle; IC=inviscid core)

Layer	Depth (km)	ρ (kg/dm ³)	ν (Pa s)	μ (GPa)
UC	0-18.5	2.65	∞	32.5
LC	18.5-28.5	2.75	$1 \cdot 10^{18}$ $0.75 \cdot 10^{19}$ $1 \cdot 10^{19}$	33.7
LCB	28.5-32.5	2.90	∞ 10^{18}	35.5
M	32.5-2891	3.39	∞ 10^{21}	73.5
IC	2891-6372	10.93	—	—

2.2 THE IRPINIA (1980) EARTHQUAKE

Comparison of measured vertical displacements from two leveling campaigns performed in 1981 and 1985 in the epicentral area of the 1980 Irpinia earthquake ($M_S = 6.9$) and predictions from viscoelastic earth models reveals the occurrence of post-seismic deformation due to stress relaxation in the ductile part of the crust, south of the area explored in the previous section 8.2.1. In proximity of the major fault, the leveling lines show a peculiar upwarping of the crust, accumulated during the time interval 1981 - 1985.

Here we test the mechanism of stress relaxation in the ductile parts of the crust after the occurrence of the shallow, normal-fault 1980 Irpinia earthquake, following the study by Dalla Via *et al.* (2003). The joint study of local and world-wide seismological data, static deformations and geological evidences provided a detailed picture of the complex mechanism of this event (Westaway and Jackson, 1984; De Natale *et al.*, 1988; Pantosti and Valensise, 1990; Pingue and De Natale, 1993). The main event consisted of three distinct subfaults at least, ruptured at intervals of about 20 s from each other. Surface faulting linked to this earthquake was well evident at several places and in particular on the main fault (first subevent), with dislocation up to 1.2 m. The total seismic moment inferred for this event is 3×10^{19} Nm.

The asymptotic expression of the fundamental solutions of the incompressible, self-gravitating, spherical, viscoelastic Earth for high harmonic degree described in Section 1.9 makes it possible, for this 1980 Irpinia application, to sum 40,000 spherical harmonic contributions in the co-seismic part and 6,000 ones in the post-seismic part, which guarantees the attainment of the highest resolution both in the co-seismic and post-seismic components. The earth model, described in Table 8.3, consists of 5 layers including a purely elastic upper crust (UC), a viscoelastic lower crust (LC), the mantle (M) and the core (IC).

The depths of the crustal layers and their elastic values have been taken from the average depths of the seismogenic crust and MOHO in the southern Apennine area (Mostardini and Merlini, 1986), while the deeper layers are based on standard global earth models. The upper limit of the lower crust, 18.5 km, has been chosen to match the maximum depths of earthquakes in this area on the basis of the assumption that the viscoelastic, ductile lower crust inhibits seismic fracture.

Viscosity in the lower crust has been varied from typical values of 10^{19} Pa s (Pollitz *et al.*, 1998). In order to mimic the reduction of the viscosity within the lower crust and the decoupling between the lower crust and the mantle, as expected on the basis of strength reduction with depth for the continental lithosphere under extension (Lynch and Morgan, 1987; Cosgrove, 1997), the bottom of the lower crust, LCB in Table 8.3, has been reduced by one order of magnitude with respect to the normal value of 10^{19} Pa s. Due to the simplified viscosity profile within the lower crust, only the effective viscosity resulting from the volumetric average within the two viscoelastic layers characterizing the lower crust can be compared with post-seismic results from other tectonic environments (Pollitz *et al.*, 1998; Pollitz *et al.*, 2000). A standard mantle (M) of 10^{21} Pa s below the lower crust does not portray any sizeable deformation over the time-scale of post-seismic deformation.

The assumed fault system consists of three normal subfaults as shown in Figure 8.9, including the three leveling lines considered, namely CZT2 (diamonds), CZT3 (crosses) and IGM81 lines (circles), measured immediately after the main shock and four years later; the thin curve in Fig 8.9, starting from Eboli and routing to Grottaminarda through Potenza and the IGM81 profile, represents the leveling line along which the co-seismic vertical displacement has been measured (Arca *et al.*, 1983; De Natale *et al.*, 1988). The surface projections of the three faults F1, F2 and F3 are shown by the light grey; the fault parameters of this model, shown in Table 8.4, are taken from Pingue and De Natale (1993). The total seismic moment has been inferred from seismological and geodetic data; for the main fault F1 the seismic moment M_0 is fixed at 24.4×10^{18} Nm, at 2.5×10^{18} Nm for F2 and at 3.2×10^{18} Nm for F3. The slip angle is fixed at -90° for each fault. Slip on the three sub-faults has been considered homogeneous in the first tests. Subsequently, by maintaining the seismic moment constant to $M_0 = 3.0 \times 10^{19}$ Nm, the slip distribution on the main fault has been varied with depth in order to reduce the misfit between model predictions and observations.

Figure 8.10 shows the observed displacements resulting from the leveling lines in Figure 8.9, where the black vertical bars reproduce the observations and their average errors; the curves represent the modeled vertical displacements due to viscoelastic relaxation without the co-seismic component, computed for various combinations of slip distribution and viscosities. In order to carry out

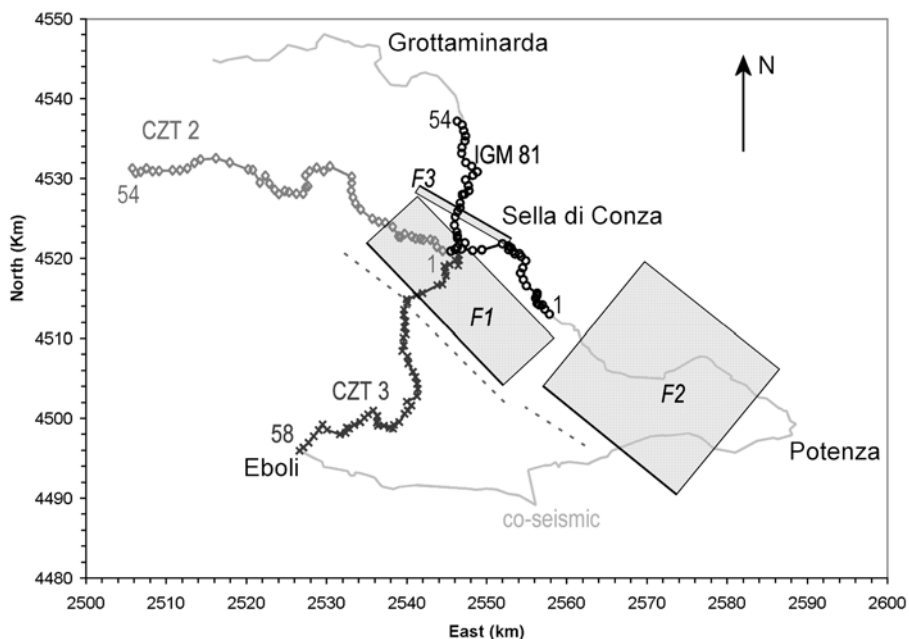


Figure 8.9. Fault model of the Irpinia 1980 earthquake. The leveling lines IGM81 (black dots), CZT2 (squares) and CZT3 (grey dots) are also indicated. F1, F2 and F3 indicate the three faults, at 0 s, 18 s and 40 s, respectively, as given in Table 8.4; the dashed lines provide the surface evidence of the faults. The first (1) and last (54, 58) benchmarks of each leveling line are also shown. Redrawn from Figure 1 in Dalla Via *et al.* (2003).

Table 8.4. L1: fault length. L2: fault width along slip direction. Top: depth of fault top margin. Disl.: mean dislocation. Str.: strike.

Sub Event	L1 (km)	L2 (km)	Top (km)	Disl. (cm)	Str. (°)	Dip (°)
F1(0s)	25	20	1.0	150	317	60
F2(18s)	22	14	1.0	25	310	20
F3(40s)	13	10	1.3	75	120	85

a comparison independent of the choice of the zero in the leveling, each model result has been uniformly shifted in such a way that the mean residual vanishes. The grey curves correspond to the reference model characterized by a uniform distribution of the seismic moment over the fault and by a lower crust viscosity, between 18.5 and 28.5 km depth, of 10^{19} Pa s (Table 8.3). Although the trends shown by this model agree with some basic features of the three lines, such

Table 8.5. Seismic moment distribution along the fault width (L2); $M=M_0/5$ where M_0 is given in the text; the fault description is given in Table 8.4.

Fault fraction	F1	F2	F3
1/5	0.8 x M	2.0 x M	2.0 x M
2/5	1.2 x M	2.0 x M	2.0 x M
3/5	1.5 x M	1.0 x M	1.0 x M
4/5	1.0 x M	0.0 x M	0.0 x M
5/5	0.5 x M	0.0 x M	0.0 x M

as the subsidence along points 1 - 20 and the following uplift of IGM81, the general uplift along CZT2 and the uplift between points 1 - 50 of CZT3, the grey curves always underestimate the observations, especially the upward bulge of 20 - 30 mm of CZT3 and the drastic increase in the uplift at point 20 of IGM81. In order to increase the upwarping of the crust along CZT3, where this line crosses the major fault, it is necessary to distribute the seismic moment in such a way that the maximum slip occurs at depths of about 10 km, as shown in Table 8.5, resulting in the dotted curves in Figure 8.10.

This best-fit seismic moment distribution has been chosen by minimizing the L^1 norm of the residuals between observations and model predictions for all the three lines simultaneously. The long-wavelength uplift between points 1 - 35 of CZT3 increases with respect to the dotted curve to 12 mm; the same is true for CZT2 for points 1 - 25, where uplift reaches 10 mm. Particularly evident is the tendency to reproduce the change from subsidence into uplift along IGM81 from the benchmarks 15 to 35. The fit between the observations and model results can be substantially improved by reducing the viscosity in the lower crust from 10^{19} Pa s to 0.75×10^{19} Pa s, as shown in Table 8.3 for the LC layer in the case of non-uniform seismic moment distribution, as indicated by the black solid curves. The increase to 23 mm in the maximum along CZT3 is accompanied by an increase to 15 mm and 10 mm along CZT2 and IGM81, respectively. The general trend of observed displacements now appears well reproduced by the best-fit model, except for some high frequency signals at very localized zones and a rather systematic underestimation of displacements in the northernmost zone of the IGM81 line.

Figure 8.11 provides an areal view of the post-seismic vertical displacement accumulated in the period 1981-1985, with characteristic zones of uplift and subsidence; this figure corresponds to the same viscosity model of Figure 8.10, characterized by lower crust relaxation (solid curves). Two major features are notable: the broadness of the area affected by subsidence and uplift and the amplitude of the displacement, of the order of 10 and 40 mm for uplift and subsidence. These effects are indicative of stress release at depth and of the

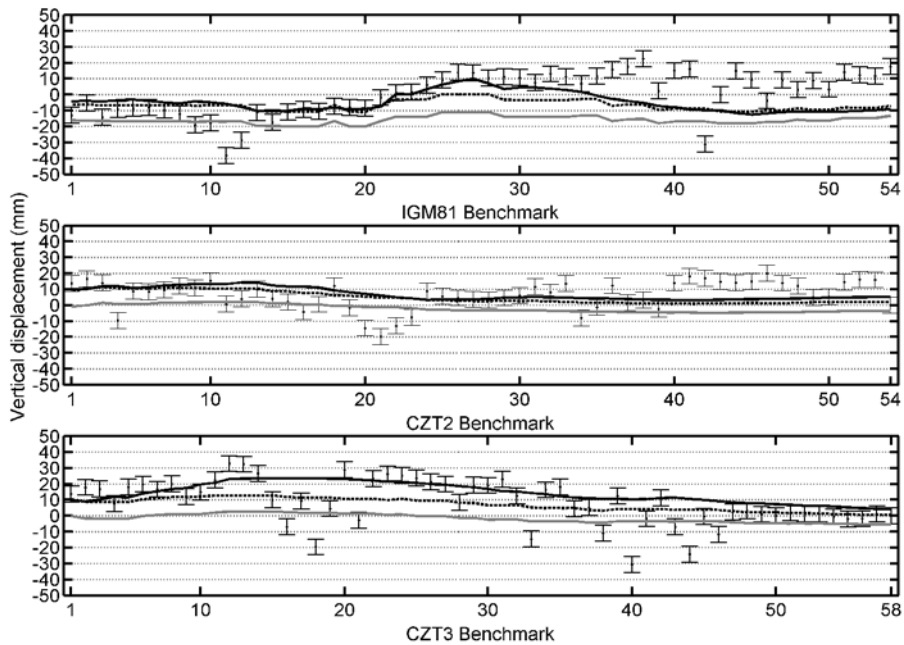


Figure 8.10. Leveling data for the three lines and modeling results: the grey curves correspond to a transition zone viscosity of 10^{19} Pa s and a uniformly distributed seismic moment, the dotted ones to the same viscosity but non-homogeneous seismic moment according to Table 8.5. The black solid curves correspond to a lower crust (LC) viscosity of 0.75×10^{19} Pa s and the seismic moment distribution in Table 8.5. Redrawn from Figure 2 in Dalla Via *et al.* (2003).

ability of the viscoelastic part of the crust to channel the flow at large distances from the fault. The expected present-day peak-to-peak post-seismic vertical displacement, 22 years after the earthquake, is 120 cm, characterized by the same pattern as in Figure 8.11.

In the case of the 1980 Irpinia earthquake area, we thus have evidence that, once averaged over the thickness of the lower crust of 14 km, the effective average viscosity of 0.6×10^{19} Pa s agrees within the order of magnitude with the ‘normal’ lower crust viscosity of about 10^{19} Pa s obtained by Pollitz *et al.* (1998) from post-seismic relaxation following the 1989 Loma Prieta earthquake.

The characteristic wavelength of the post-seismic deformation in the Irpinia area is supporting evidence for viscoelastic relaxation in the lower crust rather than in the uppermost part of the mantle, unlike the 1992 Landers earthquake where there are indications that relaxation also involves a weak upper mantle. An additional result from post-seismic displacement modeling in this area con-

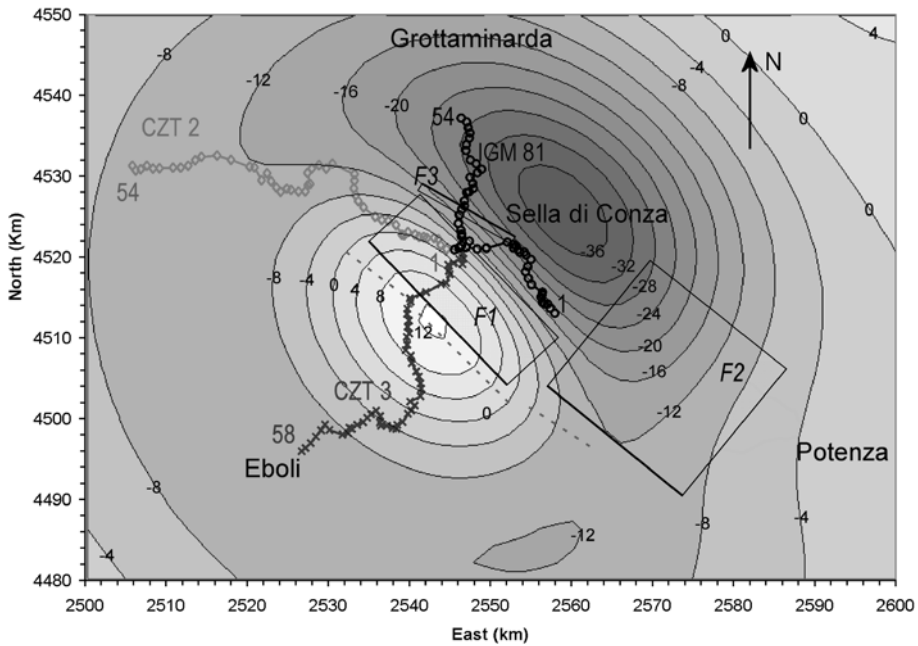


Figure 8.11. Modeling results superimposed on the leveling lines and faults in Figure 8.8, corresponding to the post-seismic displacement accumulated in the period 1981-1985 (in mm) due to crustal relaxation; fault parameters and lower crust viscosity are those corresponding to the black curves in Figure 8.9. Redrawn from the top panel of Figure 3 in Dalla Via *et al.* (2003).

sists of further details about the fault slip with respect to the previous results. It appears that higher slip concentration at around 10 km depth is able to give a best fit to the observed data.

Appendix A

Fundamental Matrix for the Compressible Case

If $j'_n(z)$ denotes differentiation with respect to the argument z of the regular (or first-order) spherical Bessel function j_n , and $y'_n(z)$ denotes differentiation with respect to the argument z of the irregular (or second-order) spherical Bessel function y_n , then the elements Y_{ij} of the fundamental matrix $\mathbf{Y}$ for the compressible case are

$$\begin{aligned}
 Y_{11} &= -\frac{1}{k^2(s)r} (n(n+1)C_k(s)j_n[k(s)r] + k(s)rj'_n[k(s)r]) \\
 Y_{12} &= -\frac{1}{k^2(s)r} ([1 + C_k(s)]j_n[k(s)r] + C_k(s)k(s)rj'_n[k(s)r]) \\
 Y_{13} &= \lambda(s)j_n[k(s)r] + 2\mu(s) \left(n(n+1) \frac{C_k(s)}{k(s)r} \left(\frac{1}{k(s)r} j_n[k(s)r] - \right. \right. \\
 &\quad \left. \left. j'_n[k(s)r] \right) - j''_n[k(s)r] \right) \\
 Y_{14} &= -\mu(s)C_k(s)j_n[k(s)r] + 2\mu(s) \left(\frac{1 + C_k(s)}{k(s)r} \left(\frac{1}{k(s)r} j_n[k(s)r] - \right. \right. \\
 &\quad \left. \left. j'_n[k(s)r] \right) - C_k(s)j''_n[k(s)r] \right) \\
 Y_{15} &= 4\pi G\rho \frac{1}{k^2(s)} j_n[k(s)r] \\
 Y_{16} &= 4\pi G\rho [1 - nC_k(s)](1+n) \frac{1}{k^2(s)r} j_n[k(s)r] \tag{A.1}
 \end{aligned}$$

$$\begin{aligned}
 Y_{21} &= -\frac{1}{q^2(s)r} (n(n+1)C_q(s)j_n[q(s)r] + q(s)rj'_n[q(s)r]) \\
 Y_{22} &= -\frac{1}{q^2(s)r} ([1 + C_q(s)]j_n[q(s)r] + C_q(s)q(s)rj'_n[q(s)r])
 \end{aligned}$$

$$\begin{aligned}
Y_{23} &= \lambda(s)j_n[q(s)r] + 2\mu(s) \left(n(n+1) \frac{C_q(s)}{q(s)r} \left(\frac{1}{q(s)r} j_n[q(s)r] - j'_n[q(s)r] \right) - j''_n[q(s)r] \right) \\
Y_{24} &= -\mu(s)C_q(s)j_n[q(s)r] + 2\mu(s) \left(\frac{1+C_q(s)}{q(s)r} \left(\frac{1}{q(s)r} j_n[q(s)r] - j'_n[q(s)r] \right) - C_q(s)j''_n[q(s)r] \right) \\
Y_{25} &= 4\pi G\rho \frac{1}{q^2(s)} j_n[q(s)r] \\
Y_{26} &= 4\pi G\rho [1 - nC_q(s)](1+n) \frac{1}{q^2(s)r} j_n[q(s)r] \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
Y_{31} &= nr^{n-1} \\
Y_{32} &= r^{n-1} \\
Y_{33} &= 2\mu(s)n(n-1)r^{n-2} \\
Y_{34} &= 2\mu(s)(n-1)r^{n-2} \\
Y_{35} &= -n\xi r^n \\
Y_{36} &= -n[(2n+1)\xi - 4\pi G\rho]r^{n-1} \tag{A.3}
\end{aligned}$$

$$\begin{aligned}
Y_{41} &= -\frac{1}{k^2(s)r} (n(n+1)C_k(s)y_n[k(s)r] + k(s)ry'_n[k(s)r]) \\
Y_{42} &= -\frac{1}{k^2(s)r} ([1 + C_k(s)]y_n[k(s)r] + C_k(s)k(s)ry'_n[k(s)r]) \\
Y_{43} &= \lambda(s)y_n[k(s)r] + 2\mu(s) \left(n(n+1) \frac{C_k(s)}{k(s)r} \left(\frac{1}{k(s)r} y_n[k(s)r] - y'_n[k(s)r] \right) - y''_n[k(s)r] \right) \\
Y_{44} &= -\mu(s)C_k(s)y_n[k(s)r] + 2\mu(s) \left(\frac{1+C_k(s)}{k(s)r} \left(\frac{1}{k(s)r} y_n[k(s)r] - y'_n[k(s)r] \right) - C_k(s)y''_n[k(s)r] \right) \\
Y_{45} &= 4\pi G\rho \frac{1}{k^2(s)} y_n[k(s)r] \\
Y_{46} &= 4\pi G\rho [1 - nC_k(s)](1+n) \frac{1}{k^2(s)r} y_n[k(s)r] \tag{A.4}
\end{aligned}$$

$$\begin{aligned}
Y_{51} &= -\frac{1}{q^2(s)r} (n(n+1)C_q(s)y_n[q(s)r] + q(s)ry'_n[q(s)r]) \\
Y_{52} &= -\frac{1}{q^2(s)r} ([1 + C_q(s)]y_n[q(s)r] + C_q(s)q(s)ry'_n[q(s)r])
\end{aligned}$$

$$\begin{aligned}
 Y_{53} &= \lambda(s)y_n[q(s)r] + 2\mu(s) \left(n(n+1) \frac{C_q(s)}{q(s)r} \left(\frac{1}{q(s)r} y_n[q(s)r] - y'_n[q(s)r] \right) - y''_n[q(s)r] \right) \\
 Y_{54} &= -\mu(s)C_q(s)y_n[q(s)r] + 2\mu(s) \left(\frac{1+C_q(s)}{q(s)r} \left(\frac{1}{q(s)r} y_n[q(s)r] - y'_n[q(s)r] \right) - C_q(s)y''_n[q(s)r] \right) \\
 Y_{55} &= 4\pi G\rho \frac{1}{q^2(s)} y_n[q(s)r] \\
 Y_{56} &= 4\pi G\rho [1 - nC_q(s)](1+n) \frac{1}{q^2(s)r} y_n[q(s)r]
 \end{aligned} \tag{A.5}$$

$$\begin{aligned}
 Y_{61} &= -\frac{n+1}{r^{n+2}} \\
 Y_{62} &= \frac{1}{r^{n+2}} \\
 Y_{63} &= 2\mu(s) \frac{(n+1)(n+2)}{r^{n+3}} \\
 Y_{64} &= -2\mu(s) \frac{n+2}{r^{n+3}} \\
 Y_{65} &= -\frac{n+1}{r^{n+1}} \xi \\
 Y_{66} &= -4\pi G\rho \frac{(n+1)}{r^{n+2}},
 \end{aligned} \tag{A.6}$$

with the wavenumbers k and q defined by

$$k^2(s) = \frac{\rho}{2}(4\pi G\rho + \xi) \frac{s + \mu/\nu}{\beta s + \kappa\mu/\nu} \left(1 + \sqrt{1 + \frac{4n(n+1)\xi^2}{(4\pi G\rho + \xi)^2} \frac{\beta s + \kappa\mu/\nu}{\mu s}} \right) \tag{A.7}$$

$$q^2(s) = \frac{\rho}{2}(4\pi G\rho + \xi) \frac{s + \mu/\nu}{\beta s + \kappa\mu/\nu} \left(1 - \sqrt{1 + \frac{4n(n+1)\xi^2}{(4\pi G\rho + \xi)^2} \frac{\beta s + \kappa\mu/\nu}{\mu s}} \right), \tag{A.8}$$

and the parameters $C_k(s)$ and $C_q(s)$ by

$$C_k(s) = -\frac{\frac{2\xi}{4\pi G\rho + \xi} \frac{\beta s + \kappa\mu/\nu}{\mu s}}{\left(1 + \sqrt{1 + \frac{4n(n+1)\xi^2}{(4\pi G\rho + \xi)^2} \frac{\beta s + \kappa\mu/\nu}{\mu s}} \right)} \tag{A.9}$$

$$C_q(s) = -\frac{\frac{2\xi}{4\pi G\rho+\xi} \frac{\beta s + \kappa\mu/\nu}{\mu s}}{\left(1 - \sqrt{1 + \frac{4n(n+1)\xi^2}{(4\pi G\rho+\xi)^2} \frac{\beta s + \kappa\mu/\nu}{\mu s}}\right)}, \quad (\text{A.10})$$

in which the bulk modulus κ and the parameter β can be expressed in the Lamé parameters λ and μ :

$$\kappa = \lambda + 2\mu/3 \quad (\text{A.11})$$

$$\beta = \lambda + 2\mu, \quad (\text{A.12})$$

and in which ξ is the quotient of the gravity g and radial distance r :

$$\xi = g/r. \quad (\text{A.13})$$

The determination of the roots of the secular equation involves the separation into five intervals, as the arguments of the spherical Bessel functions are generally complex:

- (1) $-\frac{n(n+1)\kappa\mu/\nu}{\frac{(4\pi G\rho+\xi)^2}{4\xi^2}\mu+n(n+1)\beta} < s < 0$: $k^2(s)$ and $q^2(s)$ complex, with $q^2(s)$ the complex conjugate of $k^2(s)$.
- (2) $-\frac{\kappa}{\beta}\mu/\nu < s < -\frac{n(n+1)\kappa\mu/\nu}{\frac{(4\pi G\rho+\xi)^2}{4\xi^2}\mu+n(n+1)\beta}$: $k^2(s) > 0$ and $q^2(s) > 0$.
- (3) $-\mu/\nu < s < -\frac{\kappa}{\beta}\mu/\nu$: $k^2(s) < 0$ and $q^2(s) > 0$.
- (4) $s < -\mu/\nu$: $k^2(s) > 0$ and $q^2(s) < 0$.
- (5) $s > 0$: $k^2(s) > 0$ and $q^2(s) < 0$.

In interval (1), the fundamental solutions represented by the first and second columns $\mathbf{Y}_1$ and $\mathbf{Y}_2$ of the fundamental matrix $\mathbf{Y}$ have to be combined to obtain the two new linearly independent solutions

$$\mathbf{Y}'_1 = \frac{1}{2}(\mathbf{Y}_1 + \mathbf{Y}_2) = \text{Re}(\mathbf{Y}_1) = \text{Re}(\mathbf{Y}_2) \quad (\text{A.14})$$

$$\mathbf{Y}'_2 = \frac{i}{2}(\mathbf{Y}_1 - \mathbf{Y}_2) = \text{Im}(\mathbf{Y}_1) = -\text{Im}(\mathbf{Y}_2), \quad (\text{A.15})$$

in which $i = \sqrt{-1}$. The same applies to the fourth and fifth columns $\mathbf{Y}_4$ and $\mathbf{Y}_5$.

In interval (3), the elements Y_{1j} ($j = 1, 2, 3, 4, 5, 6$) need to be changed into:

$$\begin{aligned}
 Y_{11} &= \frac{1}{h^2(s)r} (n(n+1)C_h(s)l_n[h(s)r] + h(s)r l'_n[h(s)r]) \\
 Y_{12} &= \frac{1}{h^2(s)r} ([1 + C_h(s)]l_n[h(s)r] + C_h(s)h(s)r l'_n[h(s)r]) \\
 Y_{13} &= \lambda(s)l_n[h(s)r] - 2\mu(s) \left(n(n+1) \frac{C_h(s)}{h(s)r} \left(\frac{1}{h(s)r} l_n[h(s)r] - \right. \right. \\
 &\quad \left. \left. l'_n[h(s)r] \right) - l''_n[h(s)r] \right) \\
 Y_{14} &= -\mu(s)C_h(s)l_n[h(s)r] - 2\mu(s) \left(\frac{1 + C_h(s)}{h(s)r} \left(\frac{1}{h(s)r} l_n[h(s)r] - \right. \right. \\
 &\quad \left. \left. l'_n[h(s)r] \right) - C_h(s)l''_n[h(s)r] \right) \\
 Y_{15} &= -4\pi G\rho \frac{1}{h^2(s)} l_n[h(s)r] \\
 Y_{16} &= -4\pi G\rho [1 - nC_h(s)](1+n) \frac{1}{h^2(s)r} l_n[h(s)r], \tag{A.16}
 \end{aligned}$$

with $h^2(s) = -k^2(s)$, $l_n[h(s)r]$ the spherical modified Bessel function of the first kind, and $C_h(s) = C_k(s)$.

The elements Y_{4j} ($j = 1, 2, 3, 4, 5, 6$) need to be changed into:

$$\begin{aligned}
 Y_{41} &= \frac{1}{h^2(s)r} (n(n+1)C_h(s)x_n[h(s)r] + h(s)r x'_n[h(s)r]) \\
 Y_{42} &= \frac{1}{h^2(s)r} ([1 + C_h(s)]x_n[h(s)r] + C_h(s)h(s)r x'_n[h(s)r]) \\
 Y_{43} &= \lambda(s)x_n[h(s)r] - 2\mu(s) \left(n(n+1) \frac{C_h(s)}{h(s)r} \left(\frac{1}{h(s)r} x_n[h(s)r] - \right. \right. \\
 &\quad \left. \left. x'_n[h(s)r] \right) - x''_n[h(s)r] \right) \\
 Y_{44} &= -\mu(s)C_h(s)x_n[h(s)r] - 2\mu(s) \left(\frac{1 + C_h(s)}{h(s)r} \left(\frac{1}{h(s)r} x_n[h(s)r] - \right. \right. \\
 &\quad \left. \left. x'_n[h(s)r] \right) - C_h(s)x''_n[h(s)r] \right) \\
 Y_{45} &= -4\pi G\rho \frac{1}{h^2(s)} x_n[h(s)r] \\
 Y_{46} &= -4\pi G\rho [1 - nC_h(s)](1+n) \frac{1}{h^2(s)r} x_n[h(s)r], \tag{A.17}
 \end{aligned}$$

with $x_n[h(s)r]$ the spherical modified Bessel function of the second kind.

In intervals (4) and (5), the elements Y_{2j} ($j = 1, 2, 3, 4, 5, 6$) need to be changed into:

$$\begin{aligned}
Y_{21} &= \frac{1}{p^2(s)r} (n(n+1)C_p(s)l_n[p(s)r] + p(s)r l'_n[p(s)r]) \\
Y_{22} &= \frac{1}{p^2(s)r} ([1 + C_p(s)]l_n[p(s)r] + C_p(s)p(s)r l'_n[p(s)r]) \\
Y_{23} &= \lambda(s)l_n[p(s)r] - 2\mu(s) \left(n(n+1) \frac{C_p(s)}{p(s)r} \left(\frac{1}{p(s)r} l_n[p(s)r] - \right. \right. \\
&\quad \left. \left. l'_n[p(s)r] \right) - l''_n[p(s)r] \right) \\
Y_{24} &= -\mu(s)C_p(s)l_n[p(s)r] - 2\mu(s) \left(\frac{1 + C_p(s)}{p(s)r} \left(\frac{1}{p(s)r} l_n[p(s)r] - \right. \right. \\
&\quad \left. \left. l'_n[p(s)r] \right) - C_p(s)l''_n[p(s)r] \right) \\
Y_{25} &= -4\pi G\rho \frac{1}{p^2(s)} l_n[p(s)r] \\
Y_{26} &= -4\pi G\rho [1 - nC_p(s)](1+n) \frac{1}{p(s)^2 r} l_n[p(s)r], \tag{A.18}
\end{aligned}$$

with $p^2(s) = -q^2(s)$ and $C_p(s) = C_q(s)$.

The elements Y_{5j} ($j = 1, 2, 3, 4, 5, 6$) need to be changed into:

$$\begin{aligned}
Y_{51} &= \frac{1}{p^2(s)r} (n(n+1)C_p(s)x_n[p(s)r] + p(s)r x'_n[p(s)r]) \\
Y_{52} &= \frac{1}{p^2(s)r} ([1 + C_p(s)]x_n[p(s)r] + C_p(s)p(s)r x'_n[p(s)r]) \\
Y_{53} &= \lambda(s)x_n[p(s)r] - 2\mu(s) \left(n(n+1) \frac{C_p(s)}{p(s)r} \left(\frac{1}{p(s)r} x_n[p(s)r] - \right. \right. \\
&\quad \left. \left. x'_n[p(s)r] \right) - x''_n[p(s)r] \right) \\
Y_{54} &= -\mu(s)C_p(s)x_n[p(s)r] - 2\mu(s) \left(\frac{1 + C_p(s)}{p(s)r} \left(\frac{1}{p(s)r} x_n[p(s)r] - \right. \right. \\
&\quad \left. \left. x'_n[p(s)r] \right) - C_p(s)x''_n[p(s)r] \right) \\
Y_{55} &= -4\pi G\rho \frac{1}{p^2(s)} x_n[p(s)r] \\
Y_{56} &= -4\pi G\rho [1 - nC_p(s)](1+n) \frac{1}{p^2(s)r} x_n[p(s)r]. \tag{A.19}
\end{aligned}$$

Acknowledgment: we thank Hansheng Wang for carefully checking the formulas in this Appendix.

Appendix B

Complex Contour Integration

1. ANALYTICAL FUNCTIONS

A function $f(z)$ in the complex plane is called analytical in a point $z = z_0$ if $f(z)$ is differentiable in $z = z_0$ and in a small surrounding area. Mathematically stated: the derivative

$$\frac{df}{dz} = \lim_{\delta z \rightarrow 0} \frac{\delta f}{\delta z} = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{z + \delta z - z} \quad (\text{B.1})$$

should exist. If we split the function f into real and imaginary parts as $f = u + iv$ and z into $z = x + iy$, then we get

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y} \quad (\text{B.2})$$

Setting $(\delta x \rightarrow 0, \delta y = 0)$ gives:

$$\lim_{\delta z \rightarrow 0} \frac{\delta f}{\delta z} = \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad (\text{B.3})$$

while setting $(\delta x = 0, \delta y \rightarrow 0)$ gives:

$$\lim_{\delta z \rightarrow 0} \frac{\delta f}{\delta z} = \lim_{\delta y \rightarrow 0} \left(-i \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad (\text{B.4})$$

Existence of df/dz thus leads to the following two conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (\text{B.5})$$

which are called the *Cauchy-Riemann conditions*.

The Cauchy theorem states the following: if $f(z)$ is analytical inside a region bounded by the closed contour C , then

$$\oint_C f(z)dz = 0 \quad (\text{B.6})$$

We can proof the Cauchy theorem by using the *Stokes' theorem* for converting a line integral over a closed contour into a surface integral:

$$\oint_C f(z)dz = \oint_C (u dx - v dy) + i \oint_C (v dx + u dy) \quad (\text{B.7})$$

The Gauss lemma states that

$$\oint_C f(x, y)dx = - \int_S \frac{\partial f}{\partial y} dx dy \quad (\text{B.8})$$

$$\oint_C f(x, y)dy = \int_S \frac{\partial f}{\partial x} dx dy \quad (\text{B.9})$$

where the surface S is contoured by C . Applying the Gauss lemma to equation (B.7) we obtain

$$\begin{aligned} \oint_C f(z)dz &= \int_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy \\ &+ i \int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy \end{aligned} \quad (\text{B.10})$$

Applying the Cauchy-Riemann conditions to the above equation we obtain

$$\oint_C f(z)dz = 0. \quad (\text{B.11})$$

2. CAUCHY INTEGRAL REPRESENTATION

Let us assume that $f(z)$ is an analytical function within the domain R . We can show that, if C is a closed curve in R , the following Cauchy integral representation holds

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz' \quad (\text{B.12})$$

for any z internal to C .

We consider the quantity

$$\frac{f(z') - f(z)}{z' - z} \quad (\text{B.13})$$

Since f is a continuous function, for any $\epsilon > 0$ we can find $\delta > 0$ such that for

$$|z' - z| < \delta(\epsilon) \quad (\text{B.14})$$

we have

$$|f(z') - f(z)| < \epsilon \quad (\text{B.15})$$

C denotes a circle in the z' plane centered on z of radius $r < \delta(\epsilon)$ expressed by

$$z' = z + re^{i\theta} \quad (\text{B.16})$$

The modulus of the integral of equation (B.13) over C satisfies

$$\left| \oint_C \frac{f(z') - f(z)}{z' - z} dz' \right| < \frac{\epsilon}{r} 2\pi r. \quad (\text{B.17})$$

In fact, over C we have that $|z' - z| = r$ which results into

$$\left| \oint_C \frac{f(z') - f(z)}{z' - z} dz' \right| = \left| \oint_C \frac{f(z') - f(z)}{r} dz' \right|. \quad (\text{B.18})$$

The Darboux inequality for functions of complex variables states that the above modulus of the integral over C is always smaller than the maximum of the value attained by the function which is integrated, $\frac{\epsilon}{r}$ in our case, multiplied by the length of the arc along which the function is integrated, $2\pi r$ in our case, leading to equation (B.17) from equation (B.18).

If we take the limit $\epsilon \rightarrow 0$, the right part of equation (B.17) vanishes, leading to

$$\oint_C \frac{f(z') - f(z)}{z' - z} dz' = 0 \quad (\text{B.19})$$

We thus obtain

$$\oint_C \frac{f(z')}{z' - z} dz' = f(z) \oint_C \frac{dz'}{z' - z}. \quad (\text{B.20})$$

From equation (B.16) we obtain

$$\oint_C \frac{dz'}{z' - z} = i \oint_C d\theta = 2\pi i \quad (\text{B.21})$$

that finally leads to

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z')}{z' - z} dz' \quad (\text{B.22})$$

which finally proves the result, since the curves γ and C are equivalent for the Cauchy Theorem, being obtained one from the other by continuous deformation within the domain R where $f(z)$ is analytical.

From the Cauchy integral representation of the function $f(z)$ it is straightforward to obtain the following representation of the $n - th$ derivative of $f(z)$

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z')}{(z' - z)^{n+1}} dz'. \quad (\text{B.23})$$

3. RESIDUE THEOREM

If $f(z)$ is analytical within the domain D , except for a number of isolated singularities, the Residue Theorem holds, which states that the integral of $f(z)$ extended to any closed curve γ in D , not passing through any singular point of $f(z)$, is equal to $2\pi i$ the sum of the residues of the singularities of $f(z)$ internal to γ .

Let us consider the function

$$f(z) = \frac{g(z)}{(z - z_0)^k} \quad (\text{B.24})$$

where $g(z)$ is analytical; $f(z)$ is singular with a pole of $k - th$ order in $z = z_0$. On the basis of the integral representation of the derivatives of an analytical function, we obtain

$$\oint_{\gamma} \frac{g(z)}{(z - z_0)^k} dz = \frac{2\pi i}{(k - 1)!} \frac{d^{k-1}}{dz^{k-1}} g(z) \Big|_{z=z_0}. \quad (\text{B.25})$$

By definition,

$$|Res f(z)|_{z=z_0} = \frac{1}{2\pi i} \oint_{\gamma} f(z) dz \quad (\text{B.26})$$

which gives on the basis of equation (B.23)

$$|Res f(z)|_{z=z_0} = \frac{1}{2\pi i} \oint_{\gamma} \frac{g(z)}{(z - z_0)^k} dz. \quad (\text{B.27})$$

Taking into account the Cauchy representation of the k^{th} derivative of an analytical function $g(z)$, the above expression becomes

$$|Res f(z)|_{z=z_0} = \frac{1}{(k - 1)!} \frac{d^{k-1} g(z)}{dz^{k-1}} \Big|_{z=z_0} \quad (\text{B.28})$$

or

$$|Res f(z)|_{z=z_0} = \frac{1}{(k - 1)!} \lim_{z \rightarrow z_0} \frac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z) \quad (\text{B.29})$$

If we have a simple pole in $z = z_0$, the above expression becomes

$$|Res f(z)|_{z=z_0} = \lim_{z \rightarrow z_0} (z - z_0) f(z). \quad (\text{B.30})$$

Appendix C

Forcing Terms for a Dislocation Source

1. STRIKE-SLIP

1.1 SPHEROIDAL COMPONENTS

f terms

$$f_l^{-1} = (0, 0, -\frac{2l+1}{8\pi r^3} l(l+1) \cos \alpha \frac{2l+1}{8\pi r^3} \cos \alpha, 0, 0) \quad (\text{C.1})$$

$$f_l^{-2} = (0, 0, 0, \frac{i(2l+1)}{8\pi r^3} (l-1)(l-2) \sin \alpha, 0, 0) \quad (\text{C.2})$$

$$f_l^1 = (0, 0, \frac{2l+1}{8\pi r^3} \cos \alpha, -\frac{2l+1}{8\pi r^3 l(l+1)} \cos \alpha, 0, 0) \quad (\text{C.3})$$

$$f_l^2 = (0, 0, 0, -\frac{i(2l+1)}{8\pi r^3 l(l+1)} \sin \alpha, 0, 0) \quad (\text{C.4})$$

f' terms

$$f_l'^{-1} = (0, 0, 0, \frac{(2l+1)}{8\pi r^2} \cos \alpha, 0, 0) \quad (\text{C.5})$$

$$f_l'^{-2} = (0, 0, 0, 0, 0, 0) \quad (\text{C.6})$$

$$f_l'^1 = (0, 0, 0, -\frac{(2l+1)}{8\pi r^2 l(l+1)} \cos \alpha, 0, 0) \quad (\text{C.7})$$

$$f_l'^2 = (0, 0, 0, 0, 0, 0) \quad (\text{C.8})$$

1.2 TOROIDAL COMPONENTS

f terms

$$f_l^{-1} = (0, \frac{i(2l+1)}{8\pi r^3} \cos \alpha) \quad (\text{C.9})$$

$$f_l^{-2} = (0, -\frac{(2l+1)}{8\pi r^3} (l-1)(l+2) \sin \alpha) \quad (\text{C.10})$$

$$f_l^1 = (0, \frac{i(2l+1)}{8\pi r^3 l(l+1)} \cos \alpha) \quad (\text{C.11})$$

$$f_l^2 = (0, -\frac{(2l+1)}{8\pi r^3 l(l+1)} \sin \alpha) \quad (\text{C.12})$$

f' terms

$$f_l'^{-1} = (0, \frac{i(2l+1)}{8\pi r^2} \cos \alpha) \quad (\text{C.13})$$

$$f_l'^{-2} = (0, 0) \quad (\text{C.14})$$

$$f_l'^1 = (0, \frac{i(2l+1)}{8\pi r^2 l(l+1)} \cos \alpha) \quad (\text{C.15})$$

$$f_l'^2 = (0, 0) \quad (\text{C.16})$$

2. DIP-SLIP

2.1 SPHEROIDAL COMPONENTS

f terms

$$f_l^0 = (0, 0, -\frac{(2l+1)}{4\pi r^3} \sin 2\alpha, \frac{(2l+1)}{8\pi r^3} \sin 2\alpha, 0, 0) \quad (\text{C.17})$$

$$f_l^{-1} = (0, 0, \frac{i(2l+1)}{8\pi r^3} l(l+1) \cos 2\alpha, -\frac{i(2l+1)}{8\pi r^3} \cos 2\alpha, 0, 0) \quad (\text{C.18})$$

$$f_l^1 = (0, 0, \frac{i(2l+1)}{8\pi r^3} \cos 2\alpha, -\frac{i(2l+1)}{8\pi r^3 l(l+1)} \cos 2\alpha, 0, 0) \quad (\text{C.19})$$

$$f_l^{-2} = (0, 0, 0, \frac{(2l+1)}{16\pi r^3} (l-1)(l+2) \sin 2\alpha, 0, 0) \quad (\text{C.20})$$

$$f_l^2 = (0, 0, 0, \frac{(2l+1)}{16\pi r^3 l(l+1)} \sin 2\alpha, 0, 0) \quad (\text{C.21})$$

f' terms

$$f_l'^0 = (0, 0, -\frac{(2l+1)}{4\pi r^2} \sin 2\alpha, 0, 0, 0) \quad (\text{C.22})$$

$$f_l'^{-1} = (0, 0, 0, -\frac{i(2l+1)}{8\pi r^2} \cos 2\alpha, 0, 0) \quad (\text{C.23})$$

$$f_l'^1 = (0, 0, 0, -\frac{i(2l+1)}{8\pi r^2 l(l+1)} \cos 2\alpha, 0, 0) \quad (\text{C.24})$$

$$f_l'^{-2} = (0, 0, 0, 0, 0, 0) \quad (\text{C.25})$$

$$f_l'^2 = (0, 0, 0, 0, 0, 0) \quad (\text{C.26})$$

2.2 TOROIDAL COMPONENTS

f terms

$$f_l^{-1} = (0, \frac{(2l+1)}{8\pi r^3} \cos 2\alpha) \quad (\text{C.27})$$

$$f_l^{-2} = (0, \frac{i(2l+1)}{16\pi r^3 (l-1)(l+2)} \sin 2\alpha) \quad (\text{C.28})$$

$$f_l^1 = (0, -\frac{(2l+1)}{8\pi r^3 (l+1)} \cos 2\alpha) \quad (\text{C.29})$$

$$f_l^2 = (0, -\frac{i(2l+1)}{16\pi r^3 (l+1)} \sin 2\alpha) \quad (\text{C.30})$$

f' terms

$$f_l'^{-1} = (0, \frac{(2l+1)}{8\pi r^2} \cos 2\alpha) \quad (\text{C.31})$$

$$f_l'^{-2} = (0, -\frac{(2l+1)}{8\pi r^2 l(l+1)} \cos 2\alpha) \quad (\text{C.32})$$

$$f_l'^1 = (0, 0) \quad (\text{C.33})$$

$$f_l'^{-2} = (0, 0) \quad (\text{C.34})$$

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Index

- adiabatic temperature gradient, 20
- Adriatic, 201, 202, 204–206
- Airy isostasy, 235
- angular momentum, 103
- angular velocity, 103
- Antarctica, 130, 196
- Apennines, 201, 204, 205, 279
- Apparent Polar Wander, *see* polar wander
- approximation method, *see* spherical harmonics
- Apulia, 203, 208
- APW, *see* polar wander
- Aquilaia, 203, 206, 208

- Bernese, 284
- Bessel functions, 293
- Bothnic Gulf, 190, 201, 206
- boundary conditions, 11
 - external, 19
 - internal, 18
 - internal forcing, 30
 - surface, 31, 32
- Bromwich path, 33
- bulk modulus, 296
- Burgers model, 18

- Calabrian Arc, 204
- Canada, 130, 154
- Cauchy integral representation, 300
- Cauchy theorem, 299
- Cauchy-Riemann conditions, 299
- cavitation, 18, 20
- Chandler wobble, 100, 107
 - frequency, 110
- chemical boundaries, 18
- chemical stratification, 24, 156
- Clapeyron slope, 28
- CMB, *see* core-mantle boundary
- Coble creep, 2
- complex contour integration, 33, 299
 - analytical functions, 299
 - poles, 48
 - continuous poles, 51, 81, 97
 - first order, 33, 48
 - higher order, 48
- compressibility, 4, 5, 11, 50
- continental drift, 127
- convolution, 34, 191, 249, 250, 252, 254
- convoluton, 233
- core, 18, 20
- core-mantle boundary, 18, 20
 - boundary conditions, 20
 - interface matrix, 21
- Correspondence Principle, 3, 4, 6, 8, 18
- Crete, 210
- crust, 20
 - lower crust, 279
 - transition zone, 279
 - upper crust, 279

- Darboux inequality, 301
- delta function, 3, 104, 110, 193
- density stratification, 165, 170
 - core density, 21
 - PREM, 73
- differential equations
 - toroidal part, 16
- dislocation sources
 - forcing terms, 305
- displacements, 18, 32
- dynamic form factor J_2 , 114, 125
 - changes, 127, 231
- dynamic topography, 218, 223, 264

- earth models, 45
 - 111-layer models, 53
 - 30-layer models, 53
 - 45-layer models, 53
 - 5-layer model, 40
 - 56-layer models, 53

- five-layer models
 - fixed-boundary contrast, 53
 - volume-averaged, 53
 - half-space models, 8
 - PREM, 28, 269
 - density inversions, 73
- Earth rotation, 99
 - mountain building, 227
- earthquakes
 - dip-slip, 306
 - spheroidal components, 306
 - toroidal components, 307
 - Irpinia (1980), 286
 - leveling campaigns, 286
 - seismic moment, 286
 - strike-slip, 305
 - spheroidal components, 305
 - toroidal components, 306
 - Umbria-Marche (1997), 278
 - GPS campaigns, 284
- Egnatie, 203, 210
- elastic response, 34
- equation of equilibrium, 5
- equation of momentum, 5
- equations of motion, 4
- equatorial bulge, 100, 107, 108
 - readjustment time scale, 124
- erosion, 228
- Euler equation, 103, 256
- Eulerian free precession, 107
 - frequency, 106, 110
- finite-element models, 204
- fixed-boundary contrast models, 77
- fundamental solutions matrix
 - approximation method, 36
 - compressible case, 37, 293
 - incompressible case, 12, 15, 37
 - inverse, 15
 - irregular part, 35, 38
 - regular part, 38
 - rescaling procedure, 36
 - stiffness, 35, 42
- fundamental spheroidal matrix, 17
- fundamental toroidal matrix
 - incompressible case
 - inverse, 17
- Gauss lemma, 300
- Gauss theorem, 18
- geographical frame, 106, 257
- geoid, 192, 218
 - geoid anomalies, 195
 - long-wavelength components, 173
- geoid anomalies, 192, 218
- GIA, *see* Glacial Isostatic Adjustment
- GIPSY, 284
- Glacial Isostatic Adjustment, 2, 202
- global change, 173, 180
- GPS, 269
 - GPS campaigns, 280
- GRACE, 187
- gravitational constant, 6
- gravitational potential field, 5, 32, 173
- gravity, 5, 18
- gravity anomalies
 - free-air, 195
- Green functions, 3
 - gravitational potential, 191, 192
 - radial displacement, 192
- Greenland, 130
- half-space models, 204
- Heaviside function, 81, 110, 135, 193
- Himalayas, 229
- Hooke model, 18
- hot-spot reference frame, 106, 128
- Hudson Bay, 190, 196
- hydrostatic equilibrium, 250
- hydrostatic pressure, 4
- Ice Age cycles, 150
- Ice Ages, 168
- ice mass changes
 - Antarctica, 180, 183
 - Greenland, 180, 184
 - ICE-3G, 134, 195
 - present-day, 179
 - saw-tooth function, 133
- ice sheets
 - Antarctica, 134, 136, 137
 - Fennoscandia, 134, 136, 137, 190
 - Laurentide, 134, 136, 137, 154, 190
- ICE-3G, *see* ice mass changes
- incompressibility, 5, 11
- incompressible models, 24
 - spheroidal solutions, 10
 - toroidal solutions, 16
- inertia
 - moments of inertia, 105, 113
 - products of inertia, 109, 113
- inviscid core, 18, 20, 23, 40, 49, 229
- isolation functions, 81
- Istria, 206, 208
- Kelvin-Voigt model, 18, 111
- Kronecker delta function, *see* delta function
- l.o.d., *see* length of day variations
- Lamé parameters, 17, 50, 115, 296
- Laplace domain, 4, 17
- Laplace equation, 6, 11
- Laplace transform, 17
- Laplace variable, 17
- Late Cretaceous, 216

- layering, 19
- Legendre equations, 7, 10
- Legendre polynomials, 7, 8, 10, 193
 - associated, 8
- length of day variations, 101, 107, 149
- Liouville equation, 105, 107, 109
- lithosphere, 20
 - elastic, 52, 58
 - flexure, 248
 - thickness, 149, 174, 260
 - viscoelastic, 71
- loading, 29, 132
 - centrifugal forces, 5
 - earthquakes, 5, 30
 - ice sheets, 5, 127
 - internal, 30
 - subduction, 30
 - surface loading, 31
 - point mass load, 31
 - point mass loading, 31
 - tidal forcing, 5, 31
- Love numbers, 34
 - elastic limit, 40, 48, 113
 - fluid limit, 40, 48, 54, 67, 111, 251
 - geopotential, 40
 - gravitational potential, 34, 193, 212
 - isostatic, 251
 - radial displacement, 34, 45, 193, 212
 - secular Love number, 109
 - tangential displacement, 34
 - temporal evolution, 54
 - tidal Love number, 109
 - fluid limit, 123, 252
- mantle, 2, 20
- mantle convection, 1, 3, 100, 128, 130, 149, 154, 195, 196, 218, 251, 257
- mantle stratification, 147, 168, 248, 269
- mantle viscosity, 2, 130, 136, 165
 - convex, 58, 71
 - from TPW and $\dot{J}_2$, 130
 - lower mantle, 179
 - multi-branch solutions, 116, 138, 175
 - two-layer profile, 140
 - uniform, 52
 - upper mantle, 144, 174
- Mars, 260
- Maxwell model, 3, 6, 17
- Maxwell time, 2, 23
- Mediterranean, 201
- Mediterranean, 35, 201, 202
- mega-wobble, 262, 265, 266
- Mercury, 261
- Milankovitz cycle, 132
- Miocene, 230
- momentum equation, 3, 4, 11
- Moon, 54, 99, 261
- mountain building, 227, 228
 - linear root formation, 231
 - root formation histories, 234
 - topography, 231
 - uplift histories, 229
- multi-layer models, 45
- normal mode theory, 1, 45
- numerical integration, 8
- nutation, 100
- ocean function, 114, 193
- oxygen isotopes, 132
- phase changes, 26, 156
 - univariant, 28
- phase-change boundaries, 18
- planets, 99
- Pleistocene deglaciation, 192
- Pliocene, 230
- Poisson equation, 3, 5, 10
- polar shift, 106
- polar wander, 127, 234
 - Apparent Polar Wander, 128
 - terrestrial planets and Moon, 258
 - True Polar Wander, 127, 128, 149, 168, 249
 - path, 150
 - velocities, 257
- poles, *see* complex contour integration
- post-glacial rebound, *see* Glacial Isostatic Adjustment
- post-seismic deformation, 269
 - global, 269
 - shallow earthquakes, 276
- pre-stress, 4, 5
- precession, 100
- PREM, *see* earth models
- projection operator, 22
- propagator matrix, 37
- propagator matrix technique, 18
- pseudo-spectral technique, 195
- Ravenna, 202–204, 208, 210
- Rayleigh-Taylor instabilities, 34, 74
- relaxation modes, 22
 - $C0$ mode, 23, 24
 - $L0$ mode, 22, 24
 - $M0$ mode, 22, 24
 - $M0$ rotation mode, 116
 - analytical formula, 119
 - $M1$ and $M2$ modes, 22, 24, 229
 - $M1$ mode, 214, 251, 252
 - T modes, 23
 - dilatation modes, 46
 - denumerably infinite sets, 46, 49
 - total number, 22, 49
- relaxation times, 22, 24

- tidal forcing, 112
- residue theorem, 33, 302
- response factor, 28
- rheological models, 1
- rheologies, 1
 - constitutive law, 1
 - non-linear, 2
- rigidity, 2
- Roman ruins, 202, 203
- root-solving procedure, 23
 - bisection algorithm, 23, 45
 - complex numbers, 118
 - grid-spacing, 23
- rotation
 - Earth, 227
 - terrestrial planets, 227
- rotation equation
 - long-term behavior, 123
- rotation frequency, 105
- rotation number, 264
- rotation theories, 99
 - comparison, 115
 - linearized, 110
 - non-linear, 249
 - unification, 122
- rotational deformation
 - excitation functions, 106
 - forcing function, 109
 - rigid Earth, 103
- rotational number, 124, 258
- Runge-Kutta propagation, 51, 93, 257
- Satellite Laser Ranging, 173
- Scandinavia, 130
- sea-level changes, 189, 192, 195, 218
 - eustatic, 190
 - induced by polar wander, 210
 - induced by subduction, 216
 - relative sea-level changes, 194
 - self-gravitation, 189
 - third-order cycle, 215
- sea-level equation, 190
- secular determinant, 22, 48
- secular equation
 - compressible case, 296
 - incompressible case, 22
- seismic moment, 270
- self-compression, 156
- self-gravitation, 5
- slab detachment, 228
- SLR, *see* Satellite Laser Ranging
- solution vector, 21
 - spheroidal, 10
- specific heat, 28
- Spello, 284
- spherical coordinates, 8
- spherical harmonics, 6, 8
 - high-degree approximation method, 35
 - rescaling, 39
- spheroidal solutions, 7, 19
- stiffness, *see* fundamental solutions matrix
- strain rate, 6
- strain tensor, 7
- strain-stress relations, 4
- stratification, 45, 52
- stress, 2, 18
 - non-hydrostatic, 4
- stress tensor, 4
- stress-strain relations, 4
- subduction, 249
 - distribution of slabs, 222
 - single sinking slab, 219
 - slab distribution, 257
- tectonic processes, 202, 227
- thermal diffusivity, 28
- Tibetan Plateau, 229
- toroidal solutions, 8, 23, 270
 - propagation, 23
- torque, 103
- TPW, *see* polar wander
- transient timescales, 47
- transition zone
 - high-viscosity models, 25
- True Polar Wander, *see* polar wander
- Venice, 202–204, 210
- Venus, 261
- viscoelasticity, 1, 45, 99
- viscosity, 1
 - steady-state, 3
- viscous response, 34
- VLBI, 204, 269
- volume-averaged models, 77
- wavenumbers, 295

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